Efficient Search and Responsiveness Trade-offs in a Markov Chain Model of Evolution in Dynamic Environments

Amor A. Menezes[a,*], Pierre T. Kabamba[b]

[a]*California Institute for Quantitative Biosciences, University of California, Berkeley,*
2151 Berkeley Way, Berkeley, CA 94704-5230, USA
[b]*Deceased September 20, 2014. Was with the Department of Aerospace Engineering, University of Michigan,*
1320 Beal Avenue, Ann Arbor, MI 48109-2140, USA

Abstract

Motivated by the desire to study evolutionary responsiveness in fluctuating environments, and by the current interest in analyses of evolution that merge notions of fitness maximization with dynamical systems concepts such as Lyapunov functions, this paper models natural evolution with a simple stochastic dynamical system that can be represented as a Markov chain. The process maximizes fitness globally via search and has links to information and entropy. These links suggest that a possible rationale for evolution with the exponential fitness functions observed in nature is that of optimally-efficient search in a dynamic environment, which represents the quickest trade-off of prior information about the genotype search space for search effort savings after an environment perturbation. A Lyapunov function is also provided that relates the stochastic dynamical system model with search information, and the model shows that evolution is not gradient-based but dwells longer on more fit outcomes. The model further indicates that tuning the amount of selection trades off environment responsiveness with the time to reach fit outcomes, and that excessive selection causes a loss of responsiveness, a result that is validated by the literature and impacts efforts in directed evolution.

Highlights

1. Evolution in dynamic environments may be captured by a simple, tunable Markov chain.
2. The method is not gradient-based, but dwells longer in states that are more fit.
3. Derived conditions for optimal search efficiency are met in biology and in physics.
4. Resilience losses from highly tuned selection may impact directed evolution efforts.
5. A Lyapunov function links the method's fitness maximization to search information.

Keywords: Optimization, stochastic dynamical system, information, Lyapunov function, resilience

1. Introduction

1.1. Background

Evolution has long been studied as a natural optimization method [1–3] that may explain both observed phenotype (i.e., physical and behavioral trait) diversity [1, 4] and optimal phenotype adaptation to an environment [5]. The method is considered by evolutionary ecologists to maximize fitness through natural selection [6] regardless of the definition of fitness [7] or the specifics of the map [8] from genotype (i.e., heritable genetic composition) to phenotype and thence to fitness. The method's power has been experimentally harnessed by the synthetic biology technique of directed evolution [9] to produce unnatural phenotypes [10–12] using tailored selective pressures and designed environment dynamics. As an optimization method, evolution has often been found to embody or cause trade-offs [13, 14] that may be further enhanced by variations of the environment [15].

Evolution has also been viewed as a stochastic [3, 16, 17] search [18] process, with the results of stochasticity analyzed in fluctuating environments [19–21] and modeled by the related concepts of entropy [3, 22, 23] and information [24, 25]. In its stochastic form, the process of evolution has been borrowed by computer science to perform optimization in the guise of genetic algorithms [26–28] (which can be modeled by Markov chains [29]) and evolution strategies [30–33] (which can be successfully deployed in dynamic environments [34] to also obtain biological insight [35] akin to genetic algorithms [36]).

But despite the many studies and models of evolution, there continue to be 'differing views about the efficiency, or optimality, of the adaptation model' of natural selection as a primary driver of evolution [37], and there is thus a still-open question about the optimality of the evolutionary process in addition to a question about whether natural selection maximizes fitness (the recent [38] has more on the latter question). Even the meaning of fitness maximization is unclear [38]: four varieties that are regarded by biologists for a given population are (1) the equivalence between a stable genetic equilibrium and mean fitness maximization, (2) the increase in mean fitness by natural

*Corresponding author
Email address:* `amenezes@berkeley.edu` (Amor A. Menezes)

selection when not at genetic equilibrium, (3) the equivalence between a stable genetic equilibrium and the adoption of a phenotype by all organisms that maximize individual fitness, and (4) the increase in the number of organisms adopting a phenotype that maximizes individual fitness by natural selection when not at genetic equilibrium [38].

Nevertheless, to account for, as [38] puts it, 'some cases [where] evolution by natural selection has led to traits that approximately maximize fitness within a set of feasible alternatives [6, 37, 39, 40],' fresh analyses of evolution have been pursued, using inclusive fitness [41, 42] for instance. The ongoing Formal Darwinism (FD) project [43–46] calls for 'a new kind of argument...to link equations of motion on the one hand to optimization programs on the other, and a major point is that the biologist's concept of fitness maximization is not represented by concepts from dynamical systems such as Lyapunov functions and gradient functions' [45]. The FD project seeks to axiomatize fitness and link equilibrium concepts of population genetics to solution concepts of optimization programs [46]. The FD project tries to vindicate the "individual as maximizing agent analogy" [38, 45, 47] and show that phenotypes present in an equilibrium state are optimal because no other phenotype does better in that equilibrium state.

1.2. Goals, Biological Meaning of Employed Terms, and Preview of Results

This paper provides a simple stochastic dynamical system model of natural evolution that can be represented by a discrete-time homogeneous Markov chain (see Appendix A for a background on Markov chains), with the initial model motivation consisting of examining responsiveness in the immediate aftermath of an environment fluctuation. The goals of the model are to understand what happens to the evolution process in a dynamic environment rather than to determine exact phenotype outcomes, and, like the FD project, to specify the form of the fitness function as well as to investigate the effects of changes in process dynamics on understood equilibrium concepts within the context of optimization.

No constraints are imposed on the meaning of fitness in this paper, and the model is initially developed with an abstract fitness function that ascribes value to individual phenotypes. Because the population size is restricted to one at every time step with this model for analytical simplicity, and because the abstractness of the fitness function imposes no restrictions on the constituents of a fitness valuation, it is possible to interpret the results in this paper in the context of either population mean fitness (which is trivially equivalent here to the fitness of the population's sole phenotype at a time step) or individual fitness (which includes the fitness value effect of strategies pursued by a phenotype, a possibility that is allowed by the abstract fitness function). Since model extensions will lift the unity population size restriction so that the trivial equivalence described above no longer exists, an interpretation of individual fitness is a favored one. But for this paper at least, both population mean fitness and individual fitness are mathematically-plausible meanings of fitness.

The dynamics of the model, which assumes discrete generations and single reproductive events per individual per time step, embody changes caused by natural selection at each time step. Fitness maximization in this paper is also open to both dynamical interpretations debated by biologists: a process of increasing population mean fitness, or a process of increasing adoption of a phenotype that maximizes individual fitness. These interpretations will be elaborated upon with respect to the model where appropriate. Natural selection is taken to have the traditional meaning: local competition between phenotypes in a population, with phenotypic traits arising from genotypes in a way that includes environment effects [48].

Unsurprisingly, the local selection that is described by the model maximizes fitness globally via search, and the process also has links to the concepts of information and entropy, although these links are not imposed *a priori*. What is surprising is that, when determining the form of the fitness function, a possible rationale for the evolutionary process emerges. Borne out by the kinds of fitness functions that exist in nature, this rationale constitutes optimal search efficiency in a dynamic environment, and it represents the quickest trade-off of prior information about the search space for search effort savings after an environment fluctuation occurs. Moreover, a Lyapunov function exists to relate the dynamical representation of the fitness-maximizing evolution process with search information. It is further shown that this dynamical process dwells longer on more fit outcomes, whatever they may be, instead of being gradient-based.

Additionally, insights into the effects of varying levels of selection are obtainable from the model, such as a trade-off between responsiveness to a dynamic environment and the time to reach a fit outcome. A related detrimental result (which stipulates that excessive selection causes a loss of responsiveness) is shown to be corroborated by numerous literature examples. This result has implications for the fruits of directed evolution efforts. Lastly, the similarities and differences between the model's Markov chain optimization method and other methods that are used in computer science (genetic algorithms, evolution strategies, and simulated annealing [49]) are briefly described, with the model in this paper recapturing a known Markov chain Monte Carlo technique that was originally proposed to model physical phenomena [50].

2. Methods

2.1. Problem Definition

As others have done before us (e.g., [18]), let us consider evolution to be a form of stochastic search, which looks for some genotype(s) that result(s) in some desirable phenotype(s). Let the set of genotypes, X, be a finite albeit large one, consisting of genotypes x_i, $1 \leq i \leq n$. As we shall see in Section 3.3, the fact that X is fixed does not preclude any "evolution of evolvability" [51] in a dynamic environment, a concept that we take to be a change in the ability to respond to a selection process. Let the set of phenotypes be called Z. Let the genotype-phenotype mapping be denoted by z, i.e., $z : X \rightarrow Z$, which is a function that is very important to specify when determining the phenotype outcomes

of evolution, but since we are more interested in insights from a model of evolution, let us simply say that z is an unknown function that can change with time because it includes the effects of a changing environment. For notational simplicity, we will omit explicitly denoting a function's dependence on time.

Suppose that there exists some desirable phenotype z_{des} for a particular environment (possibly, but not necessarily, a phenotype that is "best" for its environment); of course, z_{des} can also change with time to reflect environment changes, and we may not know z_{des}. We will assume that it is possible to measure differences between any two phenotypes, i.e., Z is a metric space. Let the discrepancy between the phenotype that results from a genotype x, which is $z(x)$, and the desirable, possibly unknown, phenotype z_{des} be denoted by $\|z(x) - z_{des}\|$.

Since we model evolution as a stochastic search process, we consider it to produce a probability mass function ϕ_X over the set of genotypes, $\phi_X : X \rightarrow \mathbb{R}^+$, and our model will provide dynamic transition laws that cause X to be distributed according to ϕ_X. We have postulated that evolution searches for a desirable phenotype (whatever "desirable" means), so on average, the process results in

$$\mathrm{E}_{\phi_X}[\|z(x) - z_{des}\|] = 0. \tag{1}$$

Let $y(x) = \|z(x) - z_{des}\|$, so that we can rewrite the above as $\mathrm{E}_{\phi_X}[y(x)] = 0$, where y inherits the time-dependence of z. Because we do not know z, and we may not know z_{des}, $y(x)$ is effectively an unknown function for which we know an expectation.

Let us now consider phenotype fitness, however one chooses to define fitness. Let us simply say that there exists a function f that provides some positive real value for a phenotype, $f : Z \rightarrow \mathbb{R}^+$. Thus, there exists a real-valued positive fitness function F over the genotypes too, with $F : X \rightarrow \mathbb{R}^+ : x \mapsto F(x) = (f \circ z)(x) = f(z(x))$. We will determine the form that F takes later, but for now we note that F can also change with environment dynamics, because of variations in one or both of z and f.

Since evolution depends on fitness, the sought probability mass function produced by our model must also depend on fitness. Let us assume that for any $x_i \in X$, the function $\phi_X(x_i)$ is a differentiable one of the fitnesses $F(x_1), \ldots, F(x_n)$. As previously stated, our primary model motivation is to examine responsiveness to environment fluctuations, where responsiveness is equivalent to desiring that the probability mass function change when F is perturbed. Hence, we desire that for any $\phi_X(x_i)$, the following is true for all $x_j \in X$:

$$\frac{\partial \phi_X(x_i)}{\partial F(x_j)} \neq 0. \tag{2}$$

2.2. Biological Relevance of the Motivation

Biological responsiveness was first examined as resilience in the seminal work [52], and a survey of the many definitions of resilience in the literature is available in [53]. There are numerous instances of autonomous robustness as well as resilience to small and large environment fluctuations in complex natural systems. Examples include physiological regulation in multi-cellular organisms [54, 55]; group regulation in colonies of social insects [56–58]; species evolution through adaptation and natural selection [59–61]; the rebounding of complex systems from earthquakes, tsunamis, hurricanes, asteroid strikes, etc. [62, 63]; and human brain recovery from traumatic stress [64].

Responsiveness is also central to the theory of rational behavior, which is the basis for the theory that follows in Section 3. Rational behavior [65] seeks to explain how collectives that appear in nature, which have different fractions of professions (as in beehives for example), maintain an appropriate fractional distribution among the various social functions even if one of the castes is removed. Articles on the subject now incorporate evolution and natural selection [66, 67].

2.3. A Simple Markov Chain Model

For simplicity, let us begin by using an asexual reproduction process in our evolution model, which we will expand at the end of Section 2.4 to include sexual reproduction. Accordingly, let us define a *selective evolutionary generation system* as follows.

Definition 1. A selective evolutionary generation system is a quintuple $\Gamma = (X, R, P, G, F)$, where

- X is a set of genotypes, $X = \{x_1, x_2, \ldots, x_n\}$;

- R is a set of resources whose elements can be utilized to transition between genotypes, $R = \{r_1, r_2, \ldots, r_m\}$;

- $P : R \rightarrow (0, 1]$ is a probability mass function on R representing the probability of choosing a resource at a time step, $P(r_i) = \Pr[\mathcal{R} = r_i] = p_i$, $\sum_{k=1}^{m} p_k = 1$;

- $G : X \times R \rightarrow X$ is a generation function, a mapping from one genotype to another using a resource from R;

- $F : X \rightarrow \mathbb{R}^+$ is a positive function that evaluates genotype fitness;

- X is reachable [68] through G and R; and

- the dynamics of the system are given by

$$\mathcal{X}(t + 1) = Select(\mathcal{X}(t), G(\mathcal{X}(t), \mathcal{R}(t)), N), \tag{3}$$

where $Select : X \times X \times [0, \infty) \rightarrow X$ is a random function such that if $x_1 \in X$ and $x_2 \in X$ are any two genotypes, and $N \in [0, \infty)$ is the *level of selectivity*, then

$$Select(x_1, x_2, N) = \begin{cases} x_1 & \text{with probability} \\ & \frac{F(x_1)^N}{F(x_1)^N + F(x_2)^N}, \\ x_2 & \text{with probability} \\ & \frac{F(x_2)^N}{F(x_1)^N + F(x_2)^N}. \end{cases} \tag{4}$$

In (3), $\mathcal{X}(t)$ denotes the realization of a random genotype at time t; $\mathcal{R}(t)$ denotes the realization of a random resource at time t; $G(\mathcal{X}(t), \mathcal{R}(t))$ denotes the outcome genotype mapped from

the realized genotype at time t utilizing the resource at time t; and $X(0)$ has a known probability mass function. Implicit in $G(X(t), R(t))$ is the notion that some outcome genotypes may be "easier" to achieve in the next generation by the realized genotype $X(t)$ than others, i.e., that some genotypes are more closely related than others. Also in (3), the probability of a genotype realization at some future time given the present genotype realization is conditionally independent of the past time history of genotype realizations. Thus, the dynamics of a selective evolutionary generation system form a discrete-time homogeneous Markov chain [69].

For analytical simplicity, Definition 1 restricts the population size at the start of any time step to a constant finite number that is taken as one. A description of how to lift this population size restriction to better reflect a Darwinian natural selection process is provided in Section 4.1. It follows from the imposed restriction that the population size in the middle of a time step is two, after asexual reproduction has taken place and before selection occurs. Selection in this simple model is considered to be the usual competition between individuals in a population, with selection here defined as the outcome of competition between any two individuals. Because the population size in the middle of a time step is two in Definition 1, competition thus occurs between the population's only two members during a time step. One can think of reproduction and competition as two distinct processes that take place during a time step, with the seeming situation here of a parent competing with its offspring due solely to the imposed population size restriction. When the population size is fixed at a larger number than one, it is easy to see how this model setup is a special case of the situation where every member in the population at the start of a time step produces one offspring, and then pairwise competition occurs between randomly-selected member pairs of the expanded population (and not necessarily between a parent and its offspring).

The imposed unity population size restriction at the start of a time step makes the fitness of the realization at that time equal to the population mean fitness. Admittedly, the downside of gaining mathematical tractability in this paper by imposing the population size restriction results in a contradiction to kin selection in an inclusive fitness interpretation of fitness, and also a lack of frequency dependent selection. But this setup still allows for the standard question posed by biologists, viz., "if a trait is determined by genotypes, how does the genotype frequency evolve under natural selection?" Here, genotype evolution in the single realizations can be examined instead of genotype frequency because of the population size restriction.

Other takes on Definition 1 are also possible. For example, rather than have each x_i in X be a genotype alone, each x_i can instead reflect a genotype coupled with a strategy choice. Thus, at each time step, a realization represents a (presumably beneficial) genotype and a strategy that is pursued by that genotype. Yet another interpretation is to have each x_i be a strategy choice only, a part of which is perturbed by a resource, and the dynamic process then depicts the effects of strategy choice evolution. These alternative interpretations are analogous to the "individual as maximizing agent" philosophy described in the FD project [45, 47], and fitness now has an individual fitness meaning.

The *Select* function (which captures natural selection as detailed by the biological relevance description in Section 2.4) has a number of interesting properties, including:

- For all N,

$$\frac{\Pr[Select(x_1, x_2, N) = x_1]}{\Pr[Select(x_1, x_2, N) = x_2]} = \left(\frac{F(x_1)}{F(x_2)}\right)^N. \quad (5)$$

  That is, the ratio of the probabilities of selecting any two genotypes is equal to the ratio of their respective fitnesses raised to the power N. We call this property *local rationality*, where "rational" refers to the ratio of the probabilities and is a historical term that does not imply any agency (recall that a rational number is a ratio of integers, and that rational behavior [65] examines fractions of professions).

- For $N = 0$, the values of $F(x_1)$ and $F(x_2)$ are irrelevant. That is,

$$\Pr[Select(x_1, x_2, 0) = x_1] = 1/2, \text{ and} \quad (6)$$
$$\Pr[Select(x_1, x_2, 0) = x_2] = 1/2. \quad (7)$$

- When $N \to \infty$, if $F(x_1) > F(x_2)$ then

$$\Pr[Select(x_1, x_2, N) = x_1] \to 1. \quad (8)$$

  On the other hand, if $F(x_1) < F(x_2)$ then

$$\Pr[Select(x_1, x_2, N) = x_2] \to 1. \quad (9)$$

- If $F(x_1) = F(x_2)$ then, for all N,

$$\Pr[Select(x_1, x_2, N) = x_1] = 1/2, \text{ and} \quad (10)$$
$$\Pr[Select(x_1, x_2, N) = x_2] = 1/2. \quad (11)$$

To make the Markov chain behavior clear, let us specify the pre-selection and post-selection probabilities. First, we make explicit whether or not it is possible for a genotype to reproduce using a given resource.

Definition 2. Let $\Gamma = (X, R, P, G, F)$ be a selective evolutionary generation system. Let $x_i \in X$, $x_j \in X$ and $r_k \in R$. The *descendancy tensor*, δ, has elements

$$\delta_{ijk} = \begin{cases} 1 & \text{if } x_j = G(x_i, r_k), \ 1 \le i \le n, \ 1 \le j \le n, \\ & 1 \le k \le m; \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Hence, the descendancy tensor indicates whether it is possible to produce an offspring x_j in one step from progenitor x_i via generation function G that employs resource r_k. We can use this tensor to create a matrix that represents the conditional probability of transitioning to x_j from x_i, by utilizing the probability of choosing each available resource in R and summing over all m resources as follows.

Definition 3. For the selective evolutionary generation system $\Gamma = (X, R, P, G, F)$, the matrix $\boldsymbol{\gamma}$, called the *unselective matrix of transition probabilities*, has elements

$$\gamma_{ij} = \Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i]$$

$$= \sum_{k=1}^{m} \delta_{ijk} p_k, \ 1 \le i \le n, \ 1 \le j \le n. \tag{13}$$

This matrix is a stochastic matrix (see the short proof in [70]). We can use this matrix to produce the Markov chain's matrix of transition probabilities as follows.

Definition 4. For the selective evolutionary generation system $\Gamma = (X, R, P, G, F)$, the *matrix of transition probabilities*, $\mathbf{P}$, has elements

$$P_{ij} = \Pr[\mathcal{X}(t+1) = x_j \mid \mathcal{X}(t) = x_i], \tag{14}$$

$$= \begin{cases} \Pr[Select(x_i, x_j, N) = x_j \mid \mathcal{X}(t) = x_i] \\ \times \Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i], \quad \forall j \ne i, \\ \\ \Pr[Select(x_i, x_i, N) = x_i \mid \mathcal{X}(t) = x_i] \\ \times \Pr[\text{offspring is } x_i \mid \text{progenitor is } x_i] \\ + \sum_{\substack{k=1 \\ k \ne i}}^{n} \Pr[Select(x_i, x_k, N) = x_i \mid \mathcal{X}(t) = x_i] \\ \times \Pr[\text{offspring is } x_k \mid \text{progenitor is } x_i], \quad \text{if } j = i. \end{cases} \tag{15}$$

$$= \begin{cases} \dfrac{1}{1 + \left(\frac{F(x_j)}{F(x_j)}\right)^N} \gamma_{ij}, \quad \forall j \ne i, \\ \\ \gamma_{ii} + \sum_{\substack{j=1 \\ j \ne i}}^{n} \dfrac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \gamma_{ij}, \quad \text{if } j = i. \end{cases} \tag{16}$$

The matrix of transition probabilities in (16) is also a stochastic matrix (again, a short proof is in [70]).

2.4. Biological Relevance of the Model

The two central tenets of evolution are embodied in Definition 1: *undirected variation* via the generation function, G; and *natural selection* via the $Select$ function. The general formulation involving G and resources R captures mutations, recombination, inheritance and drift, and also genetic flow; here, flow is captured by unexpected perturbations of the fitness function as described in Section 2.1. The $Select$ function that is deployed is similar to the biochemical Hill function [71], which is commonly used to model the activation or repression of gene transcription in prokaryotes.

The level of selectivity, N, has a direct biological interpretation as well, which is easily discerned using one of the possible definitions of fitness. Suppose that the fitness of a genotype is measured by the total number of descendants produced over k generations, $k \ge 1$, a prolificity typically called *future reproductive value* or *fecundity* [48]. When a colony is initiated by two progenitors x_1 and x_2, the ratio of the descendant population fractions after k generations equals the ratio of the respective future reproductive values,

$$\left(\frac{F(x_1)}{F(x_2)}\right). \tag{17}$$

After k generations, the ratio of the probability of choosing, by random sampling, a descendant of x_1 to the probability of choosing a descendant of x_2 is equal to the ratio of the descendant population fractions (17). Correspondingly, let the ratio of the probability of selecting x_1 at the initial time to the probability of selecting x_2 at the initial time, (5), be identical to the ratio of the respective prolificities, (17), with $N = 1$.

Now consider the following sequence of operations.

1. Initiate a colony with two progenitors x_1 and x_2, and let descendants be produced for k generations.
2. Extract a sample from the resulting population. Use the sample to initiate a second colony, and let descendants be produced for k generations.
3. Iterate the sample and colony initiation procedure until an Nth colony is produced.

Then, the ratio of the probability of selecting a descendant of x_1 to the probability of selecting a descendant of x_2 after using this multi-step process becomes

$$\left(\frac{F(x_1)}{F(x_2)}\right)\left(\frac{F(x_1)}{F(x_2)}\right)\cdots\left(\frac{F(x_1)}{F(x_2)}\right) = \left(\frac{F(x_1)}{F(x_2)}\right)^N, \tag{18}$$

and it is now clear that N represents the number of selections that are made, assuming a k-generation fecundity interpretation of fitness. That is, one interpretation of N is that of a parameter that is tuned during the process of directed evolution earlier described; two biological experiments that explicitly refer to this multi-selection model are [72] and [73].

Although biologists do not include mutants in determining future reproductive value because mutations are rare among descendants in static environments, the inclusion of mutants has no significant effect on this number given their rarity, and employing this finer meaning of fecundity here results in a number that remains consistent with biological practice. It is known that 'in certain...conditions, the rate of mutation increases dramatically' by as much as a million-fold [74], and employing a notion of fecundity that accounts for mutants becomes important for analyses in dynamic environments where mutations intuitively play a greater role in the evolutionary process.

The fact that nature also utilizes sexual reproduction pairs does not invalidate this model because half of the reproductive pairing can be viewed as a genotype x in the model, and the other half can be viewed as a resource r; thus, the sets of X and R have a non-empty intersection. Further, the objection that more than one resource is required for reproduction can also be included in the model without significant changes: each resource can itself be considered a set containing the water, nutrients, etc. required by a genotype for reproduction. A more involved model that thoroughly addresses these issues is deferred to future work.

3. Results

We organize the insights obtained from the preceding Markov chain model into four main results on: (1) optimization; (2) search efficiency in a dynamic environment, fitness function determination and Lyapunov function characterization; (3)

dynamic environment responsiveness and associated trade-offs; and (4) gradient ascent dissimilarity. A fifth result touches upon the model's relationship to existing literature.

3.1. Result 1: The Markov Chain Model is Included in the Class of Markov Chains That "Behave Rationally"

We first define a broader class of Markov chains and then show that our Markov chain model of the preceding section belongs to this class. Because the following sections have results that are pertinent to this entire Markov chain class, they apply to our model too.

Let $(X, \mathbf{P})$ be a time-homogeneous, irreducible, ergodic Markov chain, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of states of a Markov process, $\mathbf{P} \in \mathbb{R}^{n \times n}$ is the matrix of transition probabilities for these states, and $n < \infty$ is the number of states. Assume that the initial probability distribution over the states is known, i.e., we are given an n-vector $\mathbf{p}(0)$ having elements $p_i(0) = \Pr[\mathcal{X}(0) = x_i]$ for all $x_i \in X$, where $\mathcal{X}(0)$ denotes the state realization at time 0, and we have $\sum_{i=1}^{n} p_i(0) = 1$. Since we have assumed that the states in X are ergodic and irreducible, they admit a unique stationary probability distribution [69, 75]. Let $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_n \end{bmatrix}$ be the row vector of these stationary probabilities, satisfying the constraints $\pi_i > 0 \ \forall i$, and $\sum_{i=1}^{n} \pi_i = 1$. Let $F : X \to \mathbb{R}^+$ be a positive fitness function. Let $N \in [0, \infty)$.

Definition 5. The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is said to *behave rationally* with respect to fitness F with level N if

$$\frac{\pi_i}{\pi_j} = \left(\frac{F(x_i)}{F(x_j)} \right)^N, \ 1 \le i \le n, \ 1 \le j \le n. \tag{19}$$

This is a definition of *global rationality*, where "rational" again refers to the ratio of the probabilities (again, as in rational numbers, without any implications of agency), and "global" refers to the stationarity of these probabilities.

Each stationary probability can be explicitly characterized to ensure Markov chain rational behavior.

Theorem 1. *The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ behaves rationally with respect to fitness F with level N if and only if*

$$\pi_i = \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}, \ 1 \le i \le n. \tag{20}$$

Proof. See Appendix B. □

Here, we have a more general, probabilistic version of the optimization of an objective function. A Markov chain that behaves rationally selects the state of maximum fitness with the highest stationary probability, and, in the limit as N approaches ∞, this probability is 1. That is, N tunes the concentration of the stationary probability distribution around the state of maximum fitness, and in the limit as N approaches ∞, the problem and

solution then revert to one of standard, off-line optimization, yielding a delta function at the location of a state in X.

We now show that our model of evolution can be sufficient for Markov chain rational behavior.

Theorem 2. *For the ergodic selective evolutionary generation system $\Gamma = (X, R, P, G, F)$, assume that $\boldsymbol{\gamma}$ is symmetric. Then the Markov chain representing the stochastic dynamics of the ergodic selective evolutionary generation system behaves rationally with fitness F and level N. That is, the row vector $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_n \end{bmatrix}$, where π_i satisfies (20), is a left eigenvector of $\mathbf{P}$, the matrix of transition probabilities for Γ, with corresponding eigenvalue 1 (i.e., $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$). Hence, $\boldsymbol{\pi}$ is the vector of stationary probabilities for the selective evolutionary generation system.*

Proof. See Appendix B. □

The symmetry condition on $\boldsymbol{\gamma}$ implies that there exists equiprobable forward and reverse transitions between any pair of genotypes prior to the selection process. Theorem 2 proves that our model's underlying time-homogenous, irreducible, ergodic Markov chain possesses a stationary distribution that corresponds to Markov chain rational behavior. Convergence to this distribution is automatically guaranteed by the standard result in Markov chain theory (Appendix A) that proves convergence to a unique stationary probability distribution for a time-homogeneous, irreducible, ergodic Markov chain in the limit as time goes to infinity. Since this standard result is also independent of the initial probability distribution over X, correctness of our model as an optimization technique also follows. An alternative correctness proof that links to search efficiency is available in Theorem 4.

The requirement that mutations be reversible is satisfied in biology as *true back mutations* [76, 77], although these mutations are infrequently recorded. A minor consequence of the symmetry condition on $\boldsymbol{\gamma}$ is the following.

Theorem 3. *For the ergodic selective evolutionary generation system $\Gamma = (X, R, P, G, F)$, assume that $\boldsymbol{\gamma}$ is symmetric. Then the Markov chain representing the stochastic dynamics of the ergodic selective evolutionary generation system is time-reversible, i.e.,*

$$\pi_i P_{ij} = \pi_j P_{ji}, \ \forall i, j. \tag{21}$$

Proof. See Appendix B. □

Therefore, the Markov chain representing the stochastic dynamics of the selective evolutionary generation system and its time reversed form are statistically the same.

3.2. Result 2: Markov Chains That "Behave Rationally" Perform Search-Based Optimization Efficiently With Exponential Fitness Functions

In our initial study of Markov chain rational behavior [78], we showed that it minimizes a cross-entropy function to yield search entropy. That is, the stationary distribution $\boldsymbol{\pi}$ of the

ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N solves the optimization problem

$$\min_{\pi_1,\ldots,\pi_n} U(\boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \qquad (22)$$

subject to the constraints $\sum_{i=1}^{n} \pi_i = 1$, and $\pi_i > 0$, $\forall i$, utilizing the (normalized) fitness distribution

$$\varphi_i = \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}, \ 1 \le i \le n. \qquad (23)$$

This result states that at the optimum, the stationary distribution agrees with the fitness distribution, i.e., $\boldsymbol{\pi} = \boldsymbol{\varphi}$. A corollary is that the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ behaves rationally with respect to fitness F with level N if and only if its stationary probability distribution minimizes the "fitness-expectation of information" (the right hand side of (22), with information as defined by [79]), a so-called Principle of Least-Informative Fitness-Expectation (LIFE). At the optimum, this fitness-expectation of information is the entropy of the fitness distribution:

$$U^* = H(\boldsymbol{\varphi}) = -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i). \qquad (24)$$

Thereafter in [78], the maximization of this search entropy is investigated, based on results about efficient search from [80] and [81] that specify entropy maximization to eliminate search biases. Such search biases can be induced by, for example, predisposition of the optimization process; this predisposition causes inefficient search when it is incorrect as a result of internal or external change. Thus, to be search-efficient in a dynamic environment, an algorithm cannot incrementally construct a hypothesis and maintain good parts of the current hypothesis, since this biases the search. A maximum entropy distribution is the only one lacking bias.

Maximizing entropy has another interpretation as a trade-off. First, it follows from (22) and (24) that

$$-\sum_{i=1}^{n} \varphi_i \ln(\pi_i) \ge -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i), \qquad (25)$$

with the equality holding if and only if $\boldsymbol{\pi} = \boldsymbol{\varphi}$. It also follows that

$$-\sum_{i=1}^{n} \varphi_i \ln\left(\frac{\pi_i}{\varphi_i}\right) \ge 0, \qquad (26)$$

with the equality holding if and only if $\boldsymbol{\pi} = \boldsymbol{\varphi}$. A non-stationary version of the left-side of this expression is

$$V(\mathbf{p}(t)) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{p_i(t)}{\varphi_i}\right). \qquad (27)$$

Reference [80] provides a relationship between search theory, (27) and a similar expression

$$W(\mathbf{p}(t)) = -\sum_{i=1}^{n} p_i(t) \ln\left(\frac{p_i(t)}{\varphi_i}\right). \qquad (28)$$

Initial values $V(\mathbf{p}(0))$ and $W(\mathbf{p}(0))$ are measures of prior information for a search. It is stated that $V(\mathbf{p}(t)) - V(\mathbf{p}(0))$ is 'the measure of the amount of prior information utilized' by the search up to time t, and $W(\mathbf{p}(t)) - W(\mathbf{p}(0))$ is 'the measure of the savings in search effort thereby achieved. The optimal [search] policy is then the one that trades off initial information for reduced search effort, as quickly as possible' [80]. This policy is proved to be one of entropy maximization, resulting in optimally efficient search.

During hypothesis-independent search-based optimization with a time-varying objective function or time-varying state fitnesses, an exponential fitness function is proved to relate Markov chain rational behavior, search entropy and optimally efficient search [78]. That is, suppose that $y : X \rightarrow \mathbb{R}$ is an unknown function for which an expected value, $\mathrm{E}[y(x)]$, is a known number $Y = 0$ in accordance with Section 2.1. Then a scheme with underlying Markov chain dynamics that behave rationally and a fitness function that is exponential solves the search problem and also maximizes the search entropy while doing so. It is shown that the normalized fitness

$$\varphi_i = \alpha e^{-\beta y(x_i)}, \ 1 \le i \le n, \qquad (29)$$

(where α and β are any constants) and the stationary distribution $\boldsymbol{\pi}$ of the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N solves the optimization problem

$$\max_{\varphi_1,\ldots,\varphi_n} \min_{\pi_1,\ldots,\pi_n} U(\boldsymbol{\varphi}, \boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \qquad (30)$$

subject to the constraint $\mathrm{E}[y(x)] = Y$.

Since y is unknown, the above result is independent of knowing (possibly dynamic) z and z_{des}. The implication is that, for the case of unknown z_{des}, any exponential function that results in maximum normalized fitness at some unknown phenotype together with a scheme that makes use of Markov chain rational behavior (for instance, our Markov chain model of evolution) guarantee efficient search-based optimization. For the case of known z_{des}, one example fitness function like

$$F(x_i) = e^{-\left(K_f(z(x_i) - z_{des})^2\right)} \qquad (31)$$

together with a scheme like our model guarantee efficient search-based optimization. The example fitness function is such that a fitness between 0 and 1 is achieved depending on how well the phenotype matches the desired: a fitness of 1 represents a perfect match, whereas a fitness of 0 signifies a poor match. The example fitness function utilizes a gain parameter, K_f, which indicates how dissimilar the desired phenotype and a high-fitness phenotype are tolerated to be. Larger gains indicate greater permissiveness of poor matches. The gain parameter is also related to the level of selectivity, N, because the latter is always used as an exponent of fitness. Hence, in the above fitness function, K_f plays a similar role to N. It can be shown that the fitness function is proportional to a Gaussian probability density function with mean equal to the desired output, variance equal to $\frac{1}{2K_f^2}$, and a constant of proportionality equal to $\sqrt{\frac{2\pi}{K_f^2}}$.

In accordance with the No Free Lunch theorems on optimization [82], a Markov chain rational behavior search-based scheme is tailored to that end of the spectrum where very little is known (a predisposition hypothesis is unsuitable) and significant search space exploration is required. Of course, nature's search-based optimization procedure in a changing fitness landscape is hypothesis-independent, and the process entails significant genotype exploration. Exponential fitness functions supporting the premise of efficient search arise in nature if one considers fecundity as the measure of fitness. Other examples include the beak depth of the Galapagos finches [83] and instances when directional selection [48] is prevalent.

Correctness of our Markov chain model of evolution as an optimization technique can be explicitly proved as follows, using a measure of the prior information for a search as a Lyapunov function.

Theorem 4. *For the ergodic selective evolutionary generation system $\Gamma = (X, R, P, G, F)$, assume that γ is symmetric. Consider the discrete-time dynamic system described by*

$$\mathbf{p}(t+1) = \mathbf{p}(t)\mathbf{P}, \tag{32}$$

where $\mathbf{P}$ is the matrix of transition probabilities for Γ, and $\mathbf{p}(t)$ is an n-dimensional row vector at time t.

1. *This discrete-time dynamic system has an invariant manifold. The manifold is the set of vectors $\mathbf{p}$ with components $p_i(t) > 0$, $1 \leq i \leq n$, and $\sum_{i=1}^{n} p_i(t) = 1$.*
2. *The manifold has an equilibrium for these dynamics, $\boldsymbol{\pi}$, with components π_i satisfying (20).*
3. *The function*

$$V(\mathbf{p}(t)) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{p_i(t)}{\varphi_i}\right), \tag{33}$$

where φ_i satisfies (23), is a Lyapunov function that establishes global asymptotic stability of the dynamic system (32) with respect to the manifold.

Proof. See Appendix B. □

The Lyapunov function represents a conservation law, the conservation of prior information for a search. Its existence is consistent with Noether's theorem [84] that associates a conservation law with a symmetry in the underlying physics, since γ has been assumed symmetric.

3.3. Result 3: Markov Chains That "Behave Rationally" Exhibit Responsiveness Trade-offs

We turn now to necessary and sufficient conditions for responsiveness, and some intriguing trade-offs that result. Responsiveness of Markov chains that behave rationally is defined as the sensitivity of the stationary distribution to changes in fitness.

Definition 6. For any time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ with a positive fitness function F for all the states in X, the *extrinsic resilience* of state x_i to changes in the fitness of state x_j, $j \neq i$, is defined as

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)}, \tag{34}$$

and the *intrinsic resilience* of state x_i to changes in its own fitness is taken to be

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)}. \tag{35}$$

We say that the Markov chain $(X, \mathbf{P})$ is responsive if $\rho_{ij} \neq 0$ for all i and j.

Since the stationary distribution $\boldsymbol{\pi}$ has the closed form expression (20) for the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N, the extrinsic and intrinsic resiliencies are

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)} = \frac{-N\pi_i\pi_j}{F(x_j)}, \ \forall j \neq i, \tag{36}$$

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)} = \frac{N\pi_i(1-\pi_i)}{F(x_i)}. \tag{37}$$

These equations also provide insight into an "evolution of evolvability" [51] in a dynamic environment, which is taken here as a change in the ability to respond to a selection process. Because the extrinsic and intrinsic resiliencies depend on fitness in such a way that their sensitivity to changes in fitness are also functions of fitness, it can be concluded that resilience itself evolves in a dynamic environment. For instance,

$$\frac{\partial \rho_{ij}}{\partial F(x_j)} = \frac{\rho_{ij}}{F(x_j)}\left(N\left(1-2\pi_j\right)-1\right), \ \forall j \neq i. \tag{38}$$

The level of selectivity has the following asymptotic effect on responsiveness.

Theorem 5. *For the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N,*

$$\rho_{ij}\Big|_{\substack{N=0 \\ j\neq i}} = \rho_{ii}\Big|_{N=0} = 0, \tag{39}$$

and

$$\lim_{\substack{N\to\infty \\ j\neq i}} \rho_{ij} = \lim_{N\to\infty} \rho_{ii} = 0. \tag{40}$$

Proof. See Appendix B. □

As a result of Theorem 5, we have quantification that standard, off-line optimization ($N \to \infty$) is non-responsive. Purely random optimization ($N = 0$) is also unresponsive.

Responsiveness is a direct outcome of Markov chain rational behavior, as stated below.

Theorem 6. *The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is responsive if the chain behaves rationally.*

Proof. See Appendix B. □

Responsiveness does not always imply Markov chain rational behavior (see [70] for a counter-example). But we can state the following instead.

Theorem 7. *Ergodicity is a necessary condition for the time-homogeneous, irreducible Markov chain* $(X, \mathbf{P})$ *to be responsive.*

Proof. See Appendix B. $\qquad\qquad\square$

Furthermore, there is a fundamental trade-off between extrinsic and intrinsic resilience that is imposed by the constraint $\sum_{i=1}^{n} \pi_i = 1$. Taking the partial derivative of this constraint with respect to the fitness of state x_i, we obtain

$$\frac{\partial \pi_i}{\partial F(x_i)} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\partial \pi_j}{\partial F(x_i)} = 0. \tag{41}$$

Note that, from (36) and (37), the extrinsic resiliencies are always negative, whereas the intrinsic resiliencies are positive. Hence, (41) implies that any change in fitness that improves a state's intrinsic resilience is at the expense of the extrinsic resilience of all other states. Similarly, any change in fitness that improves a state's extrinsic resilience is at the expense of the intrinsic resilience of another state, and the extrinsic resilience of all other states.

It is possible to control the expected amount of time to reach the fittest genotype, given a starting genotype, with the level of selectivity N. We make use of the definition in Appendix A of mean hitting time to x_j given x_i, denoted σ_{ij}.

Theorem 8. *For the ergodic selective evolutionary generation system* $\Gamma = (X, R, P, G, F)$*, assume that there exists a unique index I such that $F(x_i)$ is maximized for $i = I$. Then for all $i \neq I$,*

1. $\lim_{N \to \infty} \sigma_{iI}$ *exists, and*
2. σ_{iI} *is a strictly decreasing function of N.*

Proof. See Appendix B. $\qquad\qquad\square$

Hence, a trade-off exists between responsiveness and the expected hitting time of the genotype that optimizes fitness, with the trade-off controlled by the level of selectivity, N. That is, increasing N reduces the mean hitting time to the fittest genotype but also decreases responsiveness.

The above trade-off recalls the second No Free Lunch theorem on optimization [82]. This result states that 'if one algorithm [e.g., a traditional algorithm for off-line optimization] outperforms another [e.g., our model of evolution, where the evaluation is on both duration and responsiveness] for certain kinds of cost function dynamics [e.g., static cost functions, where responsiveness is irrelevant and N is large], then the reverse must be true on the set of all other cost function dynamics [i.e., the non-static kind].'

Nature also supports the quantification that standard optimization ($N \to \infty$) is non-responsive and non-resilient. If we assume a k-generation fecundity interpretation of fitness, then $N \to \infty$ represents an infinite number of selections made over k generations. There is much biological evidence across kingdoms

to confirm that prolonged selective breeding yields non-resilient strains and vice-versa. Specific examples include the:

1. Large-scale evolution of *E. coli* to resist a single antibiotic, which causes enhanced susceptibility to any other antimicrobial agent that then replaces that antibiotic [85].
2. Selective breeding of commercial poultry for rapid growth in a disease-free environment, which results in a compromised immune function response when fowl diseases appear [86].
3. Determination that plants in a more intense selective environment (e.g., the repeated application of very strong herbicides that persist in an environment, versus less-intense selection when pathogens remain on a host for the duration of their life cycle, versus low-intensity selection of herbivore-tolerance) display greater "costs of resistance" (i.e., reductions of yield when the selective pressures are removed), with the costs 'more often found in crops versus wild species' and with a 'greater control of genetic background increas[ing] the probability of detecting [the] costs of resistance' [87].
4. Artificial selection of *D. melanogaster* for improved resistance to endoparasitoid *A. tabida*, which reduces survival when the environment is changed to one that does not have the endoparasitoid but has high competition with other unparasitized *D. melanogaster* instead [88].
5. Similar artificial selection of least killifish *H. formosa* for improved resistance to cadmium, which reduces survival when the water temperature is slowly elevated to summer water temperatures [89].
6. Alleviation of selection to promote a different response to a changing environment: in [90], house mice selected for high nest-building behavior at room temperature consume less food when the environment is changed to one of cold temperature, but if such 46th-generation mice are mated for three generations with control mice and those 46th-generation mice similarly selected for low nest-building behavior at room temperature such that the original excessive selection effects are countered (as shown by heritability estimates and mixed next-building behavior at room temperature), and if selection for high nest-building behavior at room temperatures then proceeds on the third-generation crosses for another 10 generations, the resultant offspring now consume more food when the environment is changed to one of cold temperature.

3.4. Result 4: On Local Responsiveness and Gradient Ascent Dissimilarity of Markov Chains That "Behave Rationally"

We briefly examine a selective evolutionary generation system's response to changes in selectivity and genotype fitness. First, the probability of increasing fitness with every time step, conditioned upon knowledge of the current genotype, is

$$\Pr\left[F(X(t+1)) > F(X(t)) \mid X(t) = x_i\right]$$
$$= \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{ind}\left(F(x_j) > F(x_i)\right) P_{ij}, \tag{42}$$

where ind denotes the indicator function and satisfies $\text{ind}(True) = 1$ and $\text{ind}(False) = 0$. This conditional probability represents Wright's classic notion of climbing a peak in a rugged fitness landscape [91] given a genotype and its fitness. This conditional probability increases as N increases.

However, the unconditional probability of increasing fitness with every time step,

$$\Pr\left[F\left(\mathcal{X}(t+1)\right) > F\left(\mathcal{X}(t)\right)\right]$$
$$= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \text{ind}\left(F\left(x_j\right) > F\left(x_i\right)\right) P_{ij} \pi_i, \quad (43)$$

which is not predicated on knowing whether a genotype is fit or unfit, approaches zero in the limit as N approaches ∞. That is, the unconditional probability decreases as N increases. This (perhaps counter-intuitive) result is due to the elitist nature of the resultant selection process: the genotype with maximal fitness has a stationary probability of one, and consequently, the probability of improving fitness is correspondingly zero. To quote from Wright's seminal paper [91]: 'The effect of increased severity of selection is, of course, to increase the average level of adaptation until a new equilibrium is reached. But again this is at the expense of the field of variation of the species and reduces the chance of capture of another adaptive peak.'

Next, changing genotype fitness due to a changing environment affects elements of the matrix of transition probabilities, **P**, as follows:

$$\forall j \neq i, \frac{\partial P_{ij}}{\partial F(x_j)} = \frac{N}{F(x_j)} \left(\frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)} \right)^N} \right) P_{ij}, \quad (44)$$

$$\frac{\partial P_{ii}}{\partial F(x_j)} = \frac{-N}{F(x_j)} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)} \right)^N} P_{ij}, \quad (45)$$

$$\forall j \neq i, \frac{\partial P_{ij}}{\partial F(x_i)} = \frac{-N}{F(x_i)} \left(\frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)} \right)^N} \right) P_{ij}, \quad (46)$$

$$\frac{\partial P_{ii}}{\partial F(x_i)} = \frac{N}{F(x_i)} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)} \right)^N} P_{ij}. \quad (47)$$

In the first equation above, we see that an increase in the fitness of genotype x_j increases the probability of transitioning to that genotype from current genotype x_i by an amount that is proportional to the level of selectivity and inversely proportional to the fitness value. The second equation indicates a corresponding decrease in the probability of transitioning back to the current genotype under the same altered fitness landscape. Unlike gradient ascent optimization where the transition to another genotype would be directly proportional to the fitness value, what we have here is reminiscent of the retardation property in the original rational behavior [65]; the stochastic process "slows down" transitions in more favorable fitness conditions to take advantage of the external environment. Similar effects on the transition probabilities are suggested by the latter two equations for changes in current genotype fitness.

This recalls the first, yet well-known, mathematical treatment of random fitness landscapes [92], which encapsulated this idea but for fitnesses in a constant environment: 'As the local hill is climbed, or the local optimum is reached, the rate of finding nearby fitter variants dwindles; thus in the long term the process must wait until a long jump mutation lands on the side of some distant hill, whereafter local hill climbing recommences. ...[T]he interval before the next long jump fitter variant is found must typically more than double.' In effect, [92] describes a process that spends more time in a region of greater fitness, just as with a dynamic environment above, and [92] also describes local actions that are similar to (42). Additional works that posit a non-gradient approach for evolution dynamics in a static landscape include [93, 94].

3.5. Result 5: Relationships to Computer Science and Physics

It can be shown that the Markov chain model in this paper generalizes both the computer science (1+1)-Evolution Strategy (denoted (1+1)-ES) and the Canonical Genetic Algorithm with Fitness Proportional Selection (CGAFPS) by comparing (5) to each method's ratio of the probability of selecting a candidate genotype to the probability of selecting the genotype's offspring [70]. For the (1+1)-ES, one genotype, x_1, produces one mutated offspring genotype, x_2, and the ratio of the probability of selecting x_1 to the probability of selecting x_2 is simply

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} = \frac{\text{ind}\left(F(x_1) > F(x_2)\right)}{\text{ind}\left(F(x_2) \geq F(x_1)\right)}. \quad (48)$$

The ratio in (48) is taken to be ∞ if the denominator is zero. This ratio equals (5) when the parameter N in (5) approaches ∞. For the CGAFPS, we are interested in the probability that a genotype, x_1, of the population is chosen to be a member of the population for the next generation without experiencing crossover or mutation. We then compare this probability to the probability that an offspring of x_1 is a member of the population at the next generation. Per [70], the ratio of the probability of selecting x_1 to the probability of selecting x_2 has form

$$\frac{\Pr[x_1 \text{ is selected}]}{\Pr[x_2 \text{ is selected}]} = K \frac{F(x_1)}{F(x_2)}, \ K > 0. \quad (49)$$

Although the equation above is similar to (5), the CGAFPS ratio of selection probabilities is proportional to the fitness ratio. In (49), if $K = 1$ we obtain a particular case of (5) where $N = 1$.

The Markov chain model in this paper is related to, but is not the same as, the Metropolis algorithm underlying simulated annealing; in fact, our initial efforts [78] in Markov chain rational behavior have shown that the model in this paper is equivalent to Barker's version of the Hastings algorithm [50, 69] instead of Metropolis'. Whereas the Metropolis algorithm is optimal with respect to asymptotic variance, Barker's algorithm is optimal with respect to search efficiency when applied in dynamic environments with exponential fitness functions [78]. Thus, our model appears suitable not only for biology (i.e., for capturing evolutionary processes in fluctuating environments), but also for physics (i.e., for computing the radial distribution functions for a proton-electron plasma as proposed by Barker).

Such a connection between algorithmic evolution [17] and statistical mechanics has already been recognized for Darwinian dynamics occurring in a constant environment when modeled using an approach different from a Markov chain [95]. This model suggests relative entropy as a Lyapunov function. It is thus unsurprising that an identical concept to relative entropy emerges as a Lyapunov function in this paper for an evolutionary process in a fluctuating environment, albeit in the context of search information and through a different technical approach (that of Markov chains). As [95] indicates for a constant environment, it is possible that this paper's dynamic environment Lyapunov function holds even without assuming a symmetry in the underlying physics and its ensuing time-reversibility.

4. Summary

A simple stochastic dynamical system model of natural evolution has been proposed using a Markov chain that belongs to a class of Markov chains that "behave rationally." These Markov chains maximize fitness globally via search and are tunable depending on the level of selection. Despite ignorance of the genotype-phenotype mapping and knowledge only that some desirable phenotype must be realized on average, the links that these Markov chains have to search information and entropy provide a rationale for 'the efficiency, or optimality, of the adaptation model' of natural selection as a primary driver of evolution, which is considered a still-open question [37]. When the fitness function is exponential, as are some realizations in nature, these Markov chains achieve optimal search efficiency in a dynamic environment by trading-off prior information about the search space of genotypes for search effort savings as quickly as possible after an environment perturbation occurs. This kind of fitness function also ensures that some desirable phenotype is reached, even if what is considered desirable changes with the environment perturbation. A Lyapunov function relates the dynamical system model with search information. The model is also shown to not be gradient-based, dwelling longer on more desirable phenotypes that have greater fitness.

Trade-offs suggested by Markov chains that "behave rationally" include that of tuning selection to improve the time to reach a desirable phenotype by sacrificing responsiveness. The result that these Markov chains suggest, about losing responsiveness with excessive selection, is validated by the literature and impacts directed evolution efforts in synthetic biology. It is possible that additional (Pareto-optimal) trade-offs result when the exponential fitness function is taken to be the weighted sum of exponential functions that individually represent a different objective.

4.1. Model Limitations and Efforts to Address Them

There are several limitations of the model that is developed in this paper. These include a lack of an explicit sexual reproduction mechanism (something more than simply viewing half of a reproductive pairing as a resource) and the imposition of a population size restriction for analytical simplicity (resulting in a contradiction to kin selection in an inclusive fitness interpretation of fitness and also a lack of frequency dependent selection).

But it is possible to extend our model by removing the population size restriction. A simple extension is to have each member of an initial population initiate the Markov chain process described in our model, i.e., one may consider multiple such processes to run in parallel, with the different genotype realizations of the initial population belonging to some neighborhood of the genotypes of X. Another approach is to redefine X as a set of genotype populations, where each x_i is a population of genotypes that are "close" in some way, and then model evolution as searching for some population of desirable phenotypes. The described Markov chain model is flexible in this way, with the derived results remaining unchanged. Future work will directly tackle model limitations analytically.

Appendix A. Background on Markov Chains

We briefly review theoretical concepts of discrete-time Markov chains that are relevant to this paper, as summarized from [69, 75, 96].

Definition 7. A *Markov chain* $(X, \mathbf{P})$ is a stochastic sequence of realizations $\mathcal{X}(0), \mathcal{X}(1), \ldots, \mathcal{X}(t), \mathcal{X}(t + 1), \ldots$ of elements from a set $X = \{x_1, x_2, \ldots, x_n\}$, $n < \infty$, at times $t = 0, 1, \ldots$ that result from a process starting with an initial probability mass function defined over all $x_i \in X$, namely $\mathbf{p}(0) \in \mathbb{R}^n$ having elements $p_i(0) = \Pr[\mathcal{X}(0) = x_i]$, that has conditional probabilities that satisfy the Markov property

$$\Pr[\mathcal{X}(t + 1) = x_j \mid \mathcal{X}(t) = x_i, \ldots, \mathcal{X}(0) = x_k] =$$
$$\Pr[\mathcal{X}(t + 1) = x_j \mid \mathcal{X}(t) = x_i], \tag{A.1}$$

where $\Pr[\mathcal{X}(t + 1) = x_j \mid \mathcal{X}(t) = x_i]$ is taken to be element $P_{ij}(t)$ of the matrix $\mathbf{P}(t) \in \mathbb{R}^{n \times n}$, which is called the matrix of transition probabilities.

Definition 8. The Markov chain $(X, \mathbf{P})$ is *time-homogeneous* if $P_{ij}(t)$ is independent of t for all x_i and x_j in X. That is, in Definition 7, $\Pr[\mathcal{X}(t + 1) = x_j \mid \mathcal{X}(t) = x_i]$ is taken to be element P_{ij} of the matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$.

The Markov chain $(X, \mathbf{P})$ can be described by a discrete-time dynamic system

$$\mathbf{p}(t + 1) = \mathbf{p}(t)\mathbf{P}, \tag{A.2}$$

where $\mathbf{p}(t)$ is an n-dimensional row vector that represents the probability distribution over X at time t. The components of $\mathbf{p}(t)$ satisfy $p_i(t) > 0$, $1 \le i \le n$, and $\sum_{i=1}^{n} p_i(t) = 1$. Also,

$$\mathbf{p}(t + k) = \mathbf{p}(t)\mathbf{P}^k, k = 1, 2, \ldots. \tag{A.3}$$

Definition 9. For the time-homogeneous Markov chain $(X, \mathbf{P})$, the *period* d_i of $x_i \in X$ is

$$d_i = \gcd\{k \geq 1 \mid P_{ii}^k > 0\}, \tag{A.4}$$

where $d_i = +\infty$ if there does not exist a $k \geq 1$ such that $P_{ii}^k > 0$. We say that x_i is *aperiodic* if $d_i = 1$.

Definition 10. The set X of the time-homogeneous Markov chain $(X, \mathbf{P})$ is *closed* if, for all $x_i \in X$, $\sum_{x_j \in X} P_{ij} = 1$, (i.e., there is zero probability of a transition to any element not in X). This makes $\mathbf{P}$ a stochastic matrix. The set X of the time-homogeneous Markov chain $(X, \mathbf{P})$ is *reachable* if, for all $x_i \in X$, $x_j \in X$, there exists a $k = 1, 2, \dots$ such that $P_{ij}^k > 0$. The time-homogeneous Markov chain $(X, \mathbf{P})$ is *irreducible* if X is closed and reachable.

Definition 11. For the time-homogeneous Markov chain $(X, \mathbf{P})$, the *return time* T_j to $x_j \in X$ is

$$T_j = \inf\{t \geq 1 \mid \mathcal{X}(t) = x_j\}, \tag{A.5}$$

where $T_j = \infty$ if $\mathcal{X}(t) \neq x_j$ for all $t \geq 1$. We say that x_j is *recurrent* if $\Pr[T_j < \infty \mid \mathcal{X}(0) = x_j] = 1$ and we call x_j *positive recurrent* if $\mathrm{E}[T_j \mid \mathcal{X}(0) = x_j] < \infty$. The *hitting time* of x_j is taken to be

$$S_j = \begin{cases} T_j, & \text{if } \mathcal{X}(0) \neq x_j, \\ 0, & \text{if } \mathcal{X}(0) = x_j. \end{cases} \tag{A.6}$$

The *mean hitting time* to x_j given $x_i \in X$ is defined as

$$\sigma_{ij} = \mathrm{E}[S_j \mid \mathcal{X}(0) = x_i] = \begin{cases} \sum_{k=1}^{n} P_{ik}(\sigma_{kj} + 1), & \forall i \neq j, \\ 0, & i = j, \end{cases} \tag{A.7}$$

and we take $\boldsymbol{\sigma}_j$ to be $\begin{bmatrix} \sigma_{1j} & \sigma_{2j} & \dots & \sigma_{nj} \end{bmatrix}^{\mathrm{T}}$.

If we let $\mathbf{1} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{D}_j$ be a diagonal matrix with ones on the diagonal except one zero at position (j, j), then

$$\boldsymbol{\sigma}_j = \mathbf{D}_j(\mathbf{P}\boldsymbol{\sigma}_j + \mathbf{1}). \tag{A.8}$$

Alternatively,

$$\boldsymbol{\sigma}_j = (\mathbf{I} - \mathbf{D}_j\mathbf{P})^{-1}\mathbf{D}_j\mathbf{1}, \tag{A.9}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{I} - \mathbf{D}_j\mathbf{P}$ is non-singular.

Definition 12. The time-homogeneous Markov chain $(X, \mathbf{P})$ is *ergodic* if, for all $x_i \in X$, x_i is aperiodic and positive recurrent.

Definition 13. For the time-homogeneous Markov chain $(X, \mathbf{P})$, the row vector $\boldsymbol{\pi} \in \mathbb{R}^n$ with elements $\pi_i > 0$, $1 \leq i \leq n$, and $\sum_{i=1}^{n} \pi_i = 1$ represents a *stationary probability distribution* over X if $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. The elements π_i, $1 \leq i \leq n$, are then known as *stationary probabilities*.

When the time-homogeneous Markov chain $(X, \mathbf{P})$ is irreducible and ergodic, it can be shown that a unique row vector of stationary probabilities $\boldsymbol{\pi} > 0$ exists that satisfies

$$\pi_i = \lim_{t \to \infty} p_i(t) = \frac{1}{\mathrm{E}[T_i \mid \mathcal{X}(0) = x_i]}, 1 \leq i \leq n. \tag{A.10}$$

Appendix B. Proofs

Theorem 1.

Proof. To show that (20) implies Markov chain rational behavior, consider the ratio of any π_i to π_j, $i \neq j$, where each satisfies (20). Equation (19) follows immediately. To show that Markov chain rational behavior implies (20), we begin with $\sum_{k=1}^{n} \pi_k = 1$. Dividing both sides of the equation by π_i, we obtain $\sum_{k=1}^{n} \frac{\pi_k}{\pi_i} = \frac{1}{\pi_i}$, $1 \leq i \leq n$, which, using (19), yields $\sum_{k=1}^{n} \left(\frac{F(x_k)}{F(x_i)}\right)^N = \frac{1}{\pi_i}$, $1 \leq i \leq n$. Multiplying by $F(x_i)^N$ and solving for π_i yields (20), which completes the proof. $\square$

Theorem 2.

Proof. We directly show that the row vector $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_n \end{bmatrix}$, where π_i satisfies (20), is a left eigenvector of $\mathbf{P}$, the matrix of transition probabilities for Γ, with corresponding eigenvalue 1. If the matrix of generation probabilities, $\boldsymbol{\gamma}$, is symmetric, then $\gamma_{ij} = \gamma_{ji}$, $1 \leq i \leq n$, $1 \leq j \leq n$, or equivalently,

$$\sum_{k=1}^{m} \delta_{ijk} p_k = \sum_{k=1}^{m} \delta_{jik} p_k.$$

Consider the row vector $\mathbf{v} = \boldsymbol{\pi}\mathbf{P}$. Then

$$v_j = \sum_{i=1}^{n} \pi_i P_{ij} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \pi_i P_{ij} + \pi_j P_{jj},$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} \pi_i P_{ij} + \pi_j \left(1 - \sum_{\substack{i=1 \\ i \neq j}}^{n} P_{ji}\right),$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} \pi_i P_{ij} + \pi_j - \sum_{\substack{i=1 \\ i \neq j}}^{n} \pi_j P_{ji}.$$

From (20), (13), and (16), v_j becomes

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{F(x_i)^N}{\sum_{a=1}^{n} F(x_a)^N} \frac{F(x_j)^N}{F(x_i)^N + F(x_j)^N} \sum_{k=1}^{m} \delta_{ijk} p_k \right)$$

$$+ \pi_j$$

$$- \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{F(x_j)^N}{\sum_{a=1}^{n} F(x_a)^N} \frac{F(x_i)^N}{F(x_i)^N + F(x_j)^N} \sum_{k=1}^{m} \delta_{jik} p_k \right).$$

This reduces to π_j because $\boldsymbol{\gamma}$ is symmetric. Hence, $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. $\square$

Theorem 3.

Proof. We directly show that $\pi_i P_{ij} = \pi_j P_{ji}$ for all i and j. If the matrix of generation probabilities, $\boldsymbol{\gamma}$, is symmetric, then $\gamma_{ij} = \gamma_{ji}$, $1 \le i \le n$, $1 \le j \le n$, or equivalently,

$$\sum_{k=1}^{m} \delta_{ijk} p_k = \sum_{k=1}^{m} \delta_{jik} p_k, \ 1 \le i \le n, \ 1 \le j \le n.$$

Consider $\pi_i P_{ij}$. Using (20), (13), and (16), we obtain

$$\pi_i P_{ij} = \frac{F(x_i)^N}{\sum\limits_{a=1}^{n} F(x_a)^N} \frac{F\left(x_j\right)^N}{F(x_i)^N + F\left(x_j\right)^N} \sum_{k=1}^{m} \delta_{ijk} p_k,$$

$$= \frac{F\left(x_j\right)^N}{\sum\limits_{a=1}^{n} F(x_a)^N} \frac{F(x_i)^N}{F(x_i)^N + F\left(x_j\right)^N} \sum_{k=1}^{m} \delta_{jik} p_k,$$

$$= \pi_j P_{ji}, \ 1 \le i \le n, \ 1 \le j \le n,$$

where the second equation uses the symmetry of $\boldsymbol{\gamma}$. Hence, the Markov chain representing the stochastic dynamics of the ergodic selective evolutionary generation system is time-reversible. $\square$

Theorem 4.

Proof. We use Lyapunov's Method and the LaSalle Invariance Principle [97] to directly prove this theorem.

For the ergodic selective evolutionary generation system $\Gamma = (X, R, P, G, F)$ with a symmetric matrix of generation probabilities $\boldsymbol{\gamma}$, consider the discrete-time dynamic system described by

$$\mathbf{p}(t+1) = \mathbf{p}(t)\mathbf{P},$$

where $\mathbf{P}$ is the matrix of transition probabilities for Γ, and $\mathbf{p}(t)$ is an n-dimensional row vector at time t. Here, $\mathbf{p}(t)$ is the ergodic probability distribution over the states at time t, and therefore the components of $\mathbf{p}(t)$ satisfy $p_i(t) > 0$, $1 \le i \le n$, and $\sum_{i=1}^{n} p_i(t) = 1$. Since the Markov chain underlying the selective evolutionary generation system is ergodic and irreducible, a unique equilibrium stationary distribution for these dynamics exists, $\lim_{t\to\infty} \mathbf{p}(t) = \boldsymbol{\pi}$, with components π_i satisfying (20).

Let us define $\mathbf{q}(t) = \mathbf{p}(t) - \boldsymbol{\pi}$, so that the transformed discrete-time dynamic system,

$$\mathbf{q}(t+1) = (\mathbf{q}(t) + \boldsymbol{\pi})\,\mathbf{P} - \boldsymbol{\pi},$$

has an equilibrium at the origin. The function

$$V(\mathbf{p}(t)) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{p_i(t)}{\varphi_i}\right),$$

where φ_i satisfies (23), may be rewritten as

$$V(\mathbf{q}(t) + \boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{q_i(t) + \pi_i}{\varphi_i}\right).$$

We first check the value of this transformed candidate Lyapunov equation at the origin of the transformed system. We have

$$V(\mathbf{0} + \boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{\pi_i}{\varphi_i}\right) = -\sum_{i=1}^{n} \varphi_i \ln 1 = 0,$$

because $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Next, we have to show that $\forall \mathbf{q}(t) \neq \mathbf{0}$, $V(\mathbf{q}(t) + \boldsymbol{\pi}) > 0$. But this follows directly from (26). This is because $\forall \mathbf{q}(t) \neq \mathbf{0}$,

$$V(\mathbf{q}(t) + \boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{p_i(t)}{\varphi_i}\right),$$

which is always positive according to (26).

Now consider $\Delta V = V(\mathbf{q}(t+1) + \boldsymbol{\pi}) - V(\mathbf{q}(t) + \boldsymbol{\pi})$. In the equations that follow, we assume, without loss of generality, that the fitness value of each argument of the selective evolutionary generation system is greater than or equal to one. (After all, if there exists an i such that $0 < F(x_i) < 1$, then it is possible to find a $K \in \mathbb{R}^+$ to scale all the fitness values upward, so that for all i, $KF(x_i) \ge 1$. Define the new fitnesses $F'(x_i) = KF(x_i)$, $1 \le i \le n$ and observe that the Markov chain representation of the selective evolutionary generation system is unchanged).

$$\Delta V = V(\mathbf{p}(t+1)) - V(\mathbf{p}(t)),$$

$$= -\sum_{j=1}^{n} \varphi_j \ln\left(\frac{p_j(t+1)}{\varphi_j}\right) + \sum_{j=1}^{n} \varphi_j \ln\left(\frac{p_j(t)}{\varphi_j}\right),$$

$$= -\sum_{j=1}^{n} \varphi_j \ln\left(\frac{p_j(t+1)}{p_j(t)}\right),$$

$$= -\sum_{j=1}^{n} \varphi_j \ln\left(\frac{\sum\limits_{i=1}^{n} p_i(t) P_{ij}}{p_j(t)}\right),$$

$$= -\sum_{j=1}^{n} \varphi_j \ln\left(\frac{1}{p_j(t)} \sum_{i=1}^{n} \frac{p_i(t) F(x_j)^N}{F(x_i)^N + F(x_j)^N}\right),$$

$$= -\sum_{j=1}^{n} \varphi_j \ln\left(\frac{F(x_j)^N}{p_j(t)} \sum_{i=1}^{n} \frac{p_i(t)}{F(x_i)^N + F(x_j)^N}\right).$$

Now because we have assumed, without loss of generality, that all fitnesses are greater than or equal to one, we have

$$\frac{F(x_j)^N}{p_j(t)} \sum_{i=1}^{n} \frac{p_i(t)}{F(x_i)^N + F(x_j)^N} \ge \sum_{i=1}^{n} \frac{p_i(t)}{F(x_i)^N + F(x_j)^N}$$

$$\ge \sum_{i=1}^{n} p_i(t)$$

$$\ge 1.$$

Therefore, we obtain

$$\Delta V \le -\sum_{j=1}^{n} \varphi_j \ln 1, \ \text{or}$$

$$\Delta V \le 0.$$

13

That is, ΔV is negative semi-definite, as required by Lyapunov's method.

To apply LaSalle's Invariance Principle, we have to find $Q = \{\mathbf{q}(t) | \Delta V = 0\}$. Note that

$$\Delta V = 0 = -\sum_{j=1}^{n} \varphi_j \ln 1 = -\sum_{j=1}^{n} \varphi_j \ln\left(\sum_{i=1}^{n} P_{ji}\right),$$

which can be rewritten with Bayes' Rule as

$$\Delta V = -\sum_{j=1}^{n} \varphi_j \ln\left(\sum_{i=1}^{n} \frac{\pi_i}{\pi_j} P_{ij}\right).$$

We had previously shown that

$$\Delta V = -\sum_{j=1}^{n} \varphi_j \ln\left(\frac{\sum_{i=1}^{n} p_i(t) P_{ij}}{p_j(t)}\right).$$

Thus, $\Delta V = 0$ implies that $\pi_i = p_i(t)$, $1 \le i \le n$. But from the definition of $\mathbf{q}(t)$,
$$\pi_i = p_i(t) - q_i(t),$$
and we must have that $\Delta V = 0$ implies that $q_i(t) = 0$, $1 \le i \le n$. Therefore, the only solution of the transformed discrete-time dynamic system that can stay identically in Q is the trivial solution $\mathbf{q}(t) \equiv 0$. Hence, the origin is an asymptotically stable equilibrium for the transformed discrete-time dynamic system, and therefore, the function

$$V(\mathbf{p}(t)) = -\sum_{i=1}^{n} \varphi_i \ln\left(\frac{p_i(t)}{\varphi_i}\right),$$

is a Lyapunov function for the original system with the set of vectors $\mathbf{p}$ with components $p_i(t) > 0$, $1 \le i \le n$, and $\sum_{i=1}^{n} p_i(t) = 1$ forming an invariant manifold. Moreover, since the Lyapunov function is radially unbounded, the equilibrium is globally asymptotically stable, as claimed. $\square$

Theorem 5.

Proof. We prove both parts of this theorem directly. Consider that

$$\rho_{ij}\Big|_{N=0} = \frac{-N\pi_i\pi_j}{F(x_j)}\Bigg|_{N=0},$$

$$= \frac{-N}{F(x_j)} \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N} \frac{F(x_j)^N}{\sum_{k=1}^{n} F(x_k)^N}\Bigg|_{N=0}.$$

By substitution, $\rho_{ij}\Big|_{N=0}$ is 0. Similarly,

$$\rho_{ii}\Big|_{N=0} = \frac{N\pi_i(1-\pi_i)}{F(x_i)}\Bigg|_{N=0},$$

$$= \frac{N}{F(x_i)} \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}\left(1 - \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}\right)\Bigg|_{N=0}.$$

By substitution, $\rho_{ii}\Big|_{N=0}$ is also 0.

For the second part of the theorem, we need the following lemma.

Lemma 1. *Let $0 < \alpha < 1$. Then $\lim_{N\to\infty} N\alpha^N = 0$.*

Proof of Lemma 1.

$$\lim_{N\to\infty} N\alpha^N = \lim_{N\to\infty} \frac{N}{\alpha^{-N}},$$

$$= \lim_{N\to\infty} \frac{1}{-(\alpha)^{-N}\ln\alpha} \text{ (by L'Hôpital's rule),}$$

$$= \lim_{N\to\infty} \frac{-\alpha^N}{\ln\alpha} = 0.$$

$\square$

Let I be the index for which $F(x_i)$ is maximized, and assume that I is unique. Then,

$$\lim_{N\to\infty} \frac{F(x_j)^N}{F(x_I)^N} = 0, \ \forall j \neq I, \text{ and}$$

$$\lim_{N\to\infty} \sum_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N} = 1.$$

Consider that

$$\lim_{N\to\infty} \rho_{ij} = \lim_{N\to\infty} \frac{-N\pi_i\pi_j}{F(x_j)},$$

$$= \lim_{N\to\infty} \frac{-N}{F(x_j)} \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N} \frac{F(x_j)^N}{\sum_{k=1}^{n} F(x_k)^N},$$

$$= \lim_{N\to\infty} \frac{-N}{F(x_j)} \frac{\frac{F(x_i)^N}{F(x_I)^N}}{\sum_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N}} \frac{\frac{F(x_j)^N}{F(x_I)^N}}{\sum_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N}}.$$

Now for all $i \neq j$, where $i \neq I$ and $j \neq I$, the application of Lemma 1 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim_{N\to\infty} \rho_{ij} = 0$.

If $i = I \neq j$, then the application of Lemma 1 with $\alpha = \frac{F(x_j)}{F(x_I)}$ implies that $\lim_{N\to\infty} \rho_{ij} = 0$.

Lastly, if $i \neq j = I$, then the application of Lemma 1 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim_{N\to\infty} \rho_{ij} = 0$.

Thus, for all i and j, $\lim_{N\to\infty} \rho_{ij} = 0$.

Similarly,

$$\lim_{N\to\infty} \rho_{ii} = \lim_{N\to\infty} \frac{N\pi_i(1-\pi_i)}{F(x_i)},$$

$$= \lim_{N\to\infty} \frac{N}{F(x_i)} \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}\left(1 - \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}\right),$$

$$= \lim_{N\to\infty} \frac{N}{F(x_i)} \frac{\frac{F(x_i)^N}{F(x_I)^N}}{\sum_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N}} \frac{\sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{F(x_k)^N}{F(x_I)^N}}{\sum_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N}}.$$

If $i \neq I$, then the application of Lemma 1 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim\limits_{N\to\infty} \rho_{ii} = 0$.

If $i = I$, then we have

$$\lim_{N\to\infty} \rho_{ii} = \lim_{N\to\infty} \frac{N}{F(x_I)} \frac{\frac{F(x_I)^N}{F(x_I)^N} \sum\limits_{\substack{k=1 \\ k\neq I}}^{n} \frac{F(x_k)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N} \sum\limits_{k=1}^{n} \frac{F(x_k)^N}{F(x_I)^N}}.$$

The application of Lemma 1 with $\alpha = \frac{F(x_k)}{F(x_I)}$ a total of $n-1$ times implies that $\lim\limits_{N\to\infty} \rho_{ii} = 0$.

Thus, for all i, $\lim\limits_{N\to\infty} \rho_{ii} = 0$. This completes the proof. $\square$

Theorem 6.

Proof. To show that rational behavior implies that the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is responsive, consider (36) and (37), which hold because the stationary distribution $\boldsymbol{\pi}$ has the closed form expression (20). By Definition 5, $\pi_i > 0 \; \forall i$ since the Markov chain is ergodic, $N > 0$ since the Markov chain is selective, and $F(x_i) > 0 \; \forall i$ since the fitness function is positive. Hence, $\rho_{ij} \neq 0 \; \forall i$ and j, and $(X, \mathbf{P})$ is responsive. This completes the proof. $\square$

Theorem 7.

Proof. To show that ergodicity is a necessary condition for the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ to be responsive, suppose that the chain is not ergodic. Then the chain is either not positive recurrent (i.e., it is null recurrent or transient) or it is periodic. If the chain is not positive recurrent, then there exists a state, x_i, with zero stationary probability. Suppose now that the fitness function is perturbed such that the fitness of this state, $F(x_i)$, becomes the optimal fitness value. Since the stationary probability of x_i is zero, state x_i is never visited, and therefore never considered as the optimizer. We have $\rho_{ii} = \partial \pi_i / \partial F(x_i) = 0$, and hence $(X, \mathbf{P})$ is not responsive. If the chain is periodic, then the stationary probability distribution does not exist, and responsiveness is not defined. This completes the proof. $\square$

Theorem 8.

Proof. We first prove directly that σ_{iI} converges to a constant value for each i as N approaches ∞, before inductively showing that the value of σ_{iI} does indeed decrease with increasing N.

We begin by noting that

$$\lim_{N\to\infty} P_{ij} = \lim_{N\to\infty} \frac{1}{1 + \left(\frac{F(x_i)}{F(x_j)}\right)^N} \gamma_{ij},$$

$$= \begin{cases} \gamma_{ij}, & \text{if } F(x_i) < F(x_j), \\ 0, & \text{if } F(x_i) > F(x_j), \end{cases}$$

and

$$\lim_{N\to\infty} P_{ii} = \lim_{N\to\infty} \left(\gamma_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \gamma_{ij} \right)$$

$$= \gamma_{ii} + \sum_{\substack{j=1 \\ j\neq i \\ F(x_i)>F(x_j)}}^{n} \gamma_{ij},$$

$$= 1 - \sum_{\substack{j=1 \\ j\neq i \\ F(x_i)<F(x_j)}}^{n} \gamma_{ij}.$$

Without loss of generality, assume that the elements of X of the selective evolutionary generation system are ordered according to decreasing fitness value, so that the index $I = 1$. The matrix $\lim\limits_{N\to\infty} \mathbf{P}$ is therefore a lower triangular matrix.

We seek

$$\lim_{N\to\infty} \boldsymbol{\sigma}_1 = \lim_{N\to\infty} (\mathbf{I} - \mathbf{D}_1 \mathbf{P})^{-1} \mathbf{D}_1 \mathbf{1},$$

$$= (\mathbf{I} - \mathbf{D}_1 \lim_{N\to\infty} \mathbf{P})^{-1} \mathbf{D}_1 \mathbf{1},$$

where $(\mathbf{I} - \mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})^{-1}$ always exists due to the following.

1. $\lim\limits_{N\to\infty} \mathbf{P}$ is a lower triangular matrix with full rank. All of the lower triangular elements are non-zero.
2. $\mathbf{D}_1$ is a lower triangular matrix with rank $n-1$.
3. $(\mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})$ is a matrix with zeros in row one, and elements that are equal to $\lim\limits_{N\to\infty} \mathbf{P}$ in all other rows. Hence, $(\mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})$ has rank $n-1$. Since this matrix is the product of lower triangular matrices, it is also lower triangular.
4. $(\mathbf{I} - \mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})$ is a lower triangular matrix because it is the difference of lower triangular matrices. All lower triangular elements of this matrix are non-zero, with the matrix element $(\mathbf{I} - \mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})_{11} = 1$. Thus, $(\mathbf{I} - \mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})$ has full rank.

Since $(\mathbf{I} - \mathbf{D}_1 \lim\limits_{N\to\infty} \mathbf{P})$ is a lower triangular matrix with full rank, the equation

$$(\mathbf{I} - \mathbf{D}_1 \lim_{N\to\infty} \mathbf{P}) \lim_{N\to\infty} \boldsymbol{\sigma}_1 = \mathbf{D}_1 \mathbf{1},$$

may be solved by the iterative process of forward substitution to obtain unique constant values of $\lim\limits_{N\to\infty} \sigma_{i1}$ for each i. For instance,

$$\lim_{N\to\infty} \sigma_{11} = 0,$$

$$\lim_{N\to\infty} \sigma_{21} = \frac{1}{\gamma_{21}},$$

$$\lim_{N\to\infty} \sigma_{31} = \frac{1 + \gamma_{32}(\lim\limits_{N\to\infty} \sigma_{21})}{\gamma_{31} + \gamma_{32}} = \frac{1 + \frac{\gamma_{32}}{\gamma_{21}}}{\gamma_{31} + \gamma_{32}},$$

$$\lim_{N\to\infty} \sigma_{41} = \frac{1 + \gamma_{42}(\lim\limits_{N\to\infty} \sigma_{21}) + \gamma_{43}(\lim\limits_{N\to\infty} \sigma_{31})}{\gamma_{41} + \gamma_{42} + \gamma_{43}},$$

$$= \frac{1 + \frac{\gamma_{42}}{\gamma_{21}} + \gamma_{43}\left(\frac{1+\frac{\gamma_{32}}{\gamma_{21}}}{\gamma_{31}+\gamma_{32}}\right)}{\gamma_{41} + \gamma_{42} + \gamma_{43}},$$

and so on. This completes the convergence part of the proof.

We next use induction on the argument index to show that σ_{i1} is a strictly decreasing function of N. First, consider that

$$\sigma_{21} = \frac{\sum\limits_{\substack{k=1 \\ k\neq2}}^{n} P_{2k}(\sigma_{k1} + 1) + P_{22}}{1 - P_{22}}.$$

Hence,

$$\lim_{N\to\infty} \sigma_{21} = \lim_{N\to\infty} \frac{1 + \sum\limits_{\substack{k=1 \\ k\neq2}}^{n} P_{2k}\sigma_{k1}}{1 - P_{22}},$$

$$= \frac{\lim\limits_{N\to\infty}\left(1 + \sum\limits_{\substack{k=1 \\ k\neq2}}^{n} P_{2k}\sigma_{k1}\right)}{\lim\limits_{N\to\infty}(1 - P_{22})},$$

$$= \frac{1 + \lim\limits_{N\to\infty}\left(\sum\limits_{\substack{k=1 \\ k\neq2}}^{n} P_{2k}\sigma_{k1}\right)}{1 - (1 - \gamma_{21})},$$

$$= \frac{1}{\gamma_{21}} + \frac{1}{\gamma_{21}} \lim_{N\to\infty}\left(\sum\limits_{\substack{k=1 \\ k\neq2}}^{n} P_{2k}\sigma_{k1}\right).$$

Comparing this expression to the result that was calculated by forward substitution above, $\lim\limits_{N\to\infty} \sum\limits_{\substack{k=1 \\ k\neq2}}^{n} P_{2k}\sigma_{k1}$ must decrease to 0 as N increases. Therefore, σ_{21} decreases as N increases.

For the induction hypothesis, assume that for any $s-1$ where $2 \leq (s-1) \leq (n-1)$, we have that for all t where $2 \leq t \leq (s-1)$, the mean hitting time σ_{t1} decreases with N. We now show that σ_{s1} is a decreasing function of N.

Consider that

$$\sigma_{s1} = \frac{\sum\limits_{\substack{k=1 \\ k\neq s}}^{n} P_{sk}(\sigma_{k1} + 1) + P_{ss}}{1 - P_{ss}}.$$

Hence,

$$\lim_{N\to\infty} \sigma_{s1} = \lim_{N\to\infty} \frac{1 + \sum\limits_{\substack{k=1 \\ k\neq s}}^{n} P_{sk}\sigma_{k1}}{1 - P_{ss}},$$

$$= \frac{\lim\limits_{N\to\infty}\left(1 + \sum\limits_{\substack{k=1 \\ k\neq s}}^{n} P_{sk}\sigma_{k1}\right)}{\lim\limits_{N\to\infty}(1 - P_{ss})},$$

$$= \frac{1 + \lim\limits_{N\to\infty}\left(\sum\limits_{\substack{k=1 \\ k\neq s}}^{n} P_{sk}\sigma_{k1}\right)}{\sum\limits_{k=1}^{s-1} \gamma_{sk}},$$

$$= \frac{1 + \lim\limits_{N\to\infty}\left(\sum\limits_{k=1}^{s-1} P_{sk}\sigma_{k1} + \sum\limits_{k=s+1}^{n} P_{sk}\sigma_{k1}\right)}{\sum\limits_{k=1}^{s-1} \gamma_{sk}}.$$

Comparing this expression to the general result calculated by forward substitution, $\lim\limits_{N\to\infty} \sum\limits_{k=s+1}^{n} P_{sk}\sigma_{k1}$ must decrease to 0 as N increases. By the induction hypothesis, we have $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{s-1} P_{sk}\sigma_{k1}$ decreasing with increasing N. Therefore, σ_{s1} is a decreasing function of N.

Hence, for all i where $2 \leq i \leq n$, an increase in the level of selectivity produces a corresponding decrease in the mean hitting time to the fittest argument, σ_{i1}, with $\lim\limits_{N\to\infty} \sigma_{i1}$ approaching a unique constant value for each i. $\quad\square$

References

[1] J. Maynard Smith, Optimization theory in evolution, Annual Review of Ecology and Systematics 9 (1978) 31–56.

[2] G. A. Parker, J. Maynard Smith, Optimality theory in evolutionary biology, Nature 348 (6296) (1990) 27–33.

[3] G. Kjellstrom, Evolution as a statistical optimization algorithm, Evolutionary Theory 11 (1996) 105–117.

[4] M. Brun-Usan, M. Marin-Riera, I. Salazar-Ciudad, On the effect of phenotypic dimensionality on adaptation and optimality, Journal of Evolutionary Biology 27 (12) (2014) 2614–2628.

[5] S. H. Orzack, E. Sober, Optimality models and the test of adaptationism, American Naturalist (1994) 361–380.

[6] N. B. Davies, J. R. Krebs, S. A. West, An Introduction to Behavioural Ecology, John Wiley & Sons, 2012.

[7] A. Rosenberg, F. Bouchard, Fitness, in: E. N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, Fall 2010 Edition, Stanford University, 2010.

[8] J. A. G. M. de Visser, J. Krug, Empirical fitness landscapes and the predictability of evolution, Nature Reviews Genetics 15 (2014) 480–490.

[9] P. A. Romero, F. H. Arnold, Exploring protein fitness landscapes by directed evolution, Nature Reviews Molecular Cell Biology 10 (12) (2009) 866–876.

[10] E. M. Brustad, F. H. Arnold, Optimizing non-natural protein function with directed evolution, Current Opinion in Chemical Biology 15 (2) (2011) 201–210.

[11] K. Betül, E. A. Gaucher, Experimental evolution of protein-protein interaction networks, Biochemical Journal 453 (3) (2013) 311–319.

[12] M. G. J. de Vos, F. J. Poelwijk, S. J. Tans, Optimality in evolution: new insights from synthetic biology, Current Opinion in Biotechnology 24 (4) (2013) 797–802.

[13] R. Schuetz, N. Zamboni, M. Zampieri, M. Heinemann, U. Sauer, Multidimensional optimality of microbial metabolism, Science 336 (6081) (2012) 601–604.

[14] O. Shoval, H. Sheftel, G. Shinar, Y. Hart, O. Ramote, A. Mayo, E. Dekel, K. Kavanagh, U. Alon, Evolutionary trade-offs, Pareto optimality, and the geometry of phenotype space, Science 336 (6085) (2012) 1157–1160.

[15] F. J. Poelwijk, M. G. J. de Vos, S. J. Tans, Tradeoffs and optimality in the evolution of gene regulation, Cell 146 (3) (2011) 462–470.

[16] P. Ao, Laws in Darwinian evolutionary theory, Physics of Life Reviews 2 (2) (2005) 117–156.

[17] P. Ao, Equivalent formulations of "the equation of life", Chinese Physics B 23 (7) (2014) 070513.

[18] M. D. Preston, J. W. Pitchford, A. J. Wood, Evolutionary optimality in stochastic search problems, Journal of The Royal Society Interface 7 (50) (2010) 1301–1310.

[19] M. Thattai, A. Van Oudenaarden, Stochastic gene expression in fluctuating environments, Genetics 167 (1) (2004) 523–530.

[20] D. M. Wolf, V. V. Vazirani, A. P. Arkin, Diversity in times of adversity: probabilistic strategies in microbial survival games, Journal of Theoretical Biology 234 (2) (2005) 227–253.

[21] O. Rivoire, S. Leibler, A model for the generation and transmission of variations in evolution, Proceedings of the National Academy of Sciences 111 (19) (2014) E1940–E1949.

[22] J. Collier, Entropy in evolution, Biology and Philosophy 1 (1) (1986) 5–24.

[23] E. Chastain, A. Livnat, C. Papadimitriou, U. Vazirani, Algorithms, games, and evolution, Proceedings of the National Academy of Sciences 111 (29) (2014) 10620–10623.

[24] J. Maynard Smith, The concept of information in biology, Philosophy of Science (2000) 177–194.

[25] C. G. Bowsher, P. S. Swain, Environmental sensing, information transfer, and cellular decision-making, Current Opinion in Biotechnology 28 (2014) 149–155.

[26] D. E. Goldberg, Genetic Algorithms in Search, Optimization and Machine Learning, Addison Wesley, 1989.

[27] L. Davis (Ed.), Handbook of Genetic Algorithms, Van Nostrand Reinhold, 1991.

[28] M. Mitchell, An Introduction to Genetic Algorithms, MIT Press, 1996.

[29] G. Rudolph, Convergence analysis of canonical genetic algorithms, IEEE Transactions on Neural Networks 5 (1) (1994) 96–101.

[30] I. Rechenberg, Evolutionsstrategie: Optimierung technischer systeme nach prinzipien der biologischen evolution, Ph.D. thesis, Technical University of Berlin (1971).

[31] H.-P. Schwefel, Evolution and Optimum Seeking, Wiley, 1995.

[32] H.-G. Beyer, H.-P. Schwefel, Evolution strategies: A comprehensive introduction, Natural Computing 1 (1) (2002) 3–52.

[33] D. B. Fogel, Evolutionary Computation: Toward a New Philosophy of Machine Intelligence, 3rd Edition, Wiley, 2006.

[34] J. Branke, Evolutionary Optimization in Dynamic Environments, Kluwer Academic Publishers, 2001.

[35] J. Clune, J.-B. Mouret, H. Lipson, The evolutionary origins of modularity, Proceedings of the Royal Society B: Biological sciences 280 (1755) (2013) 20122863.

[36] N. Kashtan, E. Noor, U. Alon, Varying environments can speed up evolution, Proceedings of the National Academy of Sciences 104 (34) (2007) 13711–13716.

[37] S. H. Orzack, E. Sober, Adaptationism and Optimality, Cambridge University Press, 2001.

[38] J. Birch, Natural selection and the maximization of fitness, Biological Reviews (2015) doi: 10.1111/brv.12190.

[39] R. Dawkins, The Blind Watchmaker: Why the Evidence of Evolution Reveals a Universe without Design, W W Norton & Company, 1986.

[40] H. K. Reeve, P. W. Sherman, Adaptation and the goals of evolutionary research, Quarterly Review of Biology 68 (1) (1993) 1–32.

[41] A. Grafen, Optimization of inclusive fitness, Journal of Theoretical Biology 238 (3) (2006) 541–563.

[42] S. A. West, A. Gardner, Adaptation and inclusive fitness, Current Biology 23 (13) (2013) R577–R584.

[43] A. Grafen, Formal Darwinism, the individual-as-maximizing-agent analogy and bet-hedging, Proceedings of the Royal Society of London. Series B: Biological Sciences 266 (1421) (1999) 799–803.

[44] A. Grafen, The formal Darwinism project: a mid-term report, Journal of Evolutionary Biology 20 (4) (2007) 1243–1254.

[45] A. Grafen, The formal Darwinism project in outline, Biology and Philosophy 29 (2) (2014) 155–174.

[46] A. Grafen, The formal Darwinism project in outline: Response to commentaries, Biology and Philosophy 29 (2) (2014) 281–292.

[47] S. Okasha, C. Paternotte, Adaptation, fitness and the selection-optimality links, Biology and Philosophy 29 (2) (2014) 225–232.

[48] S. Freeman, Biological Science, 2nd Edition, Pearson Prentice Hall, 2005.

[49] S. Kirkpatrick, C. D. Gelatt, Jr., M. P. Vecchi, Optimization by simulated annealing, Science 220 (4598) (1983) 671–680.

[50] A. A. Barker, Monte Carlo calculations of the radial distribution functions for a proton-electron plasma, Australian Journal of Physics 18 (1965) 119–133.

[51] M. Pigliucci, Is evolvability evolvable?, Nature Reviews Genetics 9 (1) (2008) 75–82.

[52] C. S. Holling, Resilience and stability of ecological systems, Annual Review of Ecology and Systematics 4 (1973) 1–23.

[53] F. S. Brand, K. Jax, Focusing the meaning(s) of resilience: Resilience as a descriptive concept and a boundary object, Ecology and Society 12 (1) (2007) 23.

[54] J. D. Fernstrom, R. J. Wurtman, Brain serotonin content: Physiological regulation by plasma neutral amino acids, Science 178 (4059) (1972) 414–416.

[55] G. L. Stiles, M. G. Caron, R. J. Lefkowitz, Beta-adrenergic receptors: Biochemical mechanisms of physiological regulation, Physiological Reviews 64 (2) (1984) 661–743.

[56] G. F. Oster, E. O. Wilson, Caste and Ecology in the Social Insects, Princeton University Press, 1978.

[57] G. E. Robinson, Regulation of division of labor in insect societies, Annual Review of Entomology 37 (1992) 637–665.

[58] D. M. Gordon, The organization of work in social insect colonies, Nature 380 (6570) (1996) 121–124.

[59] C. Darwin, The Origin of Species By Means of Natural Selection, or the Preservation of Favored Races in the Struggle for Life, 6th Edition, John Murray, 1876.

[60] D. N. Reznick, C. K. Ghalambor, The population ecology of contemporary adaptations: What empirical studies reveal about the conditions that promote adaptive evolution, Genetica 112–113 (1) (2001) 183–198.

[61] A. Wagner, Robustness and Evolvability in Living Systems, Princeton University Press, 2005.

[62] L. R. Walker, R. del Morel, Primary Succession and Ecosystem Rehabilitation, Cambridge University Press, 2003.

[63] T. P. Hughes, A. H. Baird, D. R. Bellwood, M. Card, S. R. Connolly, C. Folke, R. Grosberg, O. Hoegh-Guldberg, J. B. C. Jackson, J. Kleypas, J. M. Lough, P. Marshall, M. Nyström, S. R. Palumbi, J. M. Pandolfi, B. Rosen, J. Roughgarden, Climate change, human impacts and the resilience of coral reefs, Science 301 (5635) (2003) 929–933.

[64] V. Hughes, The roots of resilience, Nature 490 (7419) (2012) 165–167.

[65] S. M. Meerkov, Mathematical theory of behavior — individual and collective behavior of retardable elements, Mathematical Biosciences 43 (1–2) (1979) 41–106.

[66] E. Bonabeau, G. Theraulaz, J.-L. Deneubourg, S. Aron, S. Camazine, Self-organization in social insects, Trends in Ecology and Evolution 12 (5) (1997) 188–193.

[67] S. A. Levin, Ecosystems and the biosphere as complex adaptive systems, Ecosystems 1 (5) (1998) 431–436.

[68] R. Diestel, Graph Theory, 3rd Edition, Springer, 2005.

[69] P. Brémaud, Markov Chains: Gibbs fields, Monte Carlo Simulation and Queues, Springer, 1999.

[70] A. A. Menezes, Selective evolutionary generation systems: Theory and applications, Ph.D. thesis, University of Michigan (2010).

[71] U. Alon, An Introduction to Systems Biology; Design Principles of Biological Circuits, CRC Press, 2007.

[72] D. Burd, Plastic not fantastic, in: Project Reports of the Canada Wide Science Fair, Waterloo-Wellington Science and Engineering Fair, 2008.

[73] W. C. Ratcliff, R. F. Denison, M. Borrello, M. Travisano, Experimental evolution of multicellularity, Proceedings of the National Academy of Sciences 109 (5) (2012) 1595–1600.

[74] S. J. Balin, M. Cascalho, The rate of mutation of a single gene, Nucleic Acids Research 38 (5) (2010) 1575–1582.

[75] C. G. Cassandras, S. Lafortune, Introduction to Discrete Event Systems, 2nd Edition, Springer, 2008.

[76] E. J. Gardner, M. J. Simmons, D. P. Snustad, Principles of Genetics, 8th Edition, Wiley, 1998.

[77] K. J. Ryan, C. G. Ray (Eds.), Sherris Medical Microbiology: An Introduction to Infectious Diseases, 4th Edition, McGraw-Hill, 2004.

[78] A. A. Menezes, P. T. Kabamba, Optimal search efficiency of Barker's algorithm with an exponential fitness function, Optimization Letters 8 (2) (2014) 691–703.

[79] C. E. Shannon, A mathematical theory of communication, Bell System Technical Journal 27 (1948) 379–423 and 623–656.

[80] E. T. Jaynes, Entropy and search theory, in: Proceedings of the First Maximum Entropy Workshop, 1981.

[81] E. T. Jaynes, Information theory and statistical mechanics, The Physical Review 106 (4) (1957) 620–630.

[82] D. H. Wolpert, W. G. Macready, No free lunch theorems for optimization, IEEE Transactions on Evolutionary Computation 1 (1) (1997) 67–82.

[83] D. Schluter, Estimating the form of natural selection on a quantitative trait, Evolution 42 (5) (1988) 849–861.

[84] E. Noether, Invariante variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse 1918 (1918) 235–257.

[85] V. Lázár, G. P. Singh, R. Spohn, I. Nagy, B. Horváth, M. Hrtyan, R. Busa-Fekete, B. Bogos, O. Méhi, B. Csörgő, G. Pósfai, G. Fekete, B. Szappanos, B. Kégl, B. Papp, C. Pál, Bacterial evolution of antibiotic hypersensitivity, Molecular Systems Biology 9 (1) (2013) 700.

[86] P. J. van der Most, B. de Jong, H. K. Parmentier, S. Verhulst, Trade-off between growth and immune function: A meta-analysis of selection experiments, Functional Ecology 25 (1) (2011) 74–80.

[87] J. Bergelson, C. B. Purrington, Surveying patterns in the cost of resistance in plants, The American Naturalist 148 (3) (1996) 536–558.

[88] A. R. Kraaijveld, H. C. J. Godfray, Trade-off between parasitoid resistance and larval competitive ability in *Drosophila Melanogaster*, Nature 389 (6648) (1997) 278–280.

[89] L. Xie, P. L. Klerks, Responses to selection for cadmium resistance in the least killifish, *Heterandria Formosa*, Environmental Toxicology and Chemistry 22 (2) (2003) 313–320.

[90] A. Bult, C. B. Lynch, Breaking through artificial selection limits of an adaptive behavior in mice and the consequences for correlated responses, Behavior Genetics 30 (3) (2000) 193–206.

[91] S. Wright, The roles of mutation, inbreeding, crossbreeding and selection in evolution, in: Proceedings of the Sixth International Congress of Genetics, Vol. 1, 1932, pp. 356–366.

[92] S. Kauffman, S. Levin, Towards a general theory of adaptive walks on rugged landscapes, Journal of Theoretical Biology 128 (1) (1987) 11–45.

[93] P. Ao, T.-Q. Chen, J.-H. Shi, Dynamical decomposition of Markov processes without detailed balance, Chinese Physics Letters 30 (7) (2013) 070201.

[94] S. Xu, S. Jiao, P. Jiang, P. Ao, Two-time-scale population evolution on a singular landscape, Physical Review E 89 (1) (2014) 012724.

[95] P. Ao, Emerging of stochastic dynamical equalities and steady state thermodynamics from Darwinian dynamics, Communications in Theoretical Physics 49 (5) (2008) 1073–1090.

[96] J. A. Gubner, Probability and Random Processes for Electrical and Computer Engineers, Cambridge University Press, 2006.

[97] H. K. Khalil, Nonlinear Systems, 3rd Edition, Prentice Hall, 2002.