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Rational Behavior Design Using Multi-Selective Generation

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Abstract— **This paper extends a technique that solves a generalization of the standard global optimization problem: instead of generating the optimizer, the technique produces, on the search space, a probability density function referred to as the behavior. The generalized solution depends on a parameter, the level of selectivity, such that as this parameter tends to infinity, the behavior becomes a delta function at the location of the optimizer. The motivation for this generalization is that traditional off-line global optimization is unresponsive to perturbations of the objective function. Although the original technique achieves responsive optimization, a large number of iterations may be required. In most instances, the extended technique of this paper, which is known as multi-selective generation, averages fewer iterations to achieve responsive optimization. Multi-selective generation is formulated here to generalize the canonical genetic algorithm with fitness proportional selection. Necessary and sufficient conditions that are required by multi-selective generation to achieve so-called rational behavior are specified. Rational behavior is desirable because it can lead to both efficient search and responsive optimization. However, the conditions for the extended technique to behave rationally are highly restrictive. The implication is that the original technique, which behaves rationally, is preferable for efficient search and responsive optimization.**

I. INTRODUCTION

A. Motivation, Goals and Contributions

THIS paper considers the problem of *behavior design*, which, for a real-valued reward function $F : D \to \mathbb{R}$, seeks 1) a probability density function (referred to as the *behavior*) $\phi_X : D \to \mathbb{R}^+$ that accomplishes specified objectives, and 2) dynamic transition laws that cause the variable x to be distributed according to ϕ_X, i.e., to exhibit the behavior specified by ϕ_X.

A well-known particular case of behavior design is *off-line optimization*, where the sought behavior consists of a delta function at the location that optimizes the reward function. Off-line optimization techniques [1], however, are *non-responsive* to perturbations of the reward function. Specifically, small changes in the reward function may require changes in the behavior when the optimizer depends continuously or discontinuously on the perturbation. Hence, the motivation for behavior design is that in practice, the reward function on which a candidate optimizer is implemented may be different from that for which the candidate's behavior was designed.

On-line optimization methods [2] that are more responsive than their off-line counterparts typically consist of the sequential repetition of off-line optimization techniques. However, such sequential repetitions are computationally expensive, a fact that may be shown by either an amortized analysis [3] or a competitive analysis [2]. Reference [4] presented a behavior design technique that is computationally inexpensive, yields behaviors that are responsive, and employs a scheme that is *efficient* in that it trades off prior information about the search space for search effort savings as quickly as possible. This technique, which utilizes a Selective Evolutionary Generation System (SEGS), was shown to differ from most other evolutionary computation approaches; in fact, the Canonical Genetic Algorithm with Fitness Proportional Selection (CGAFPS) [5] and the (1+1) evolutionary strategy [6] are particular cases of the technique.

Unfortunately, a SEGS may require a large number of steps to generate a candidate whose reward (or fitness) exceeds a threshold. Therefore, one goal of this paper is to extend the work of [4] so that responsive behavior design is inexpensively attained in fewer iterations, on average. The increased speed is particularly important for the finite-horizon problem, when a fit candidate must be found within a pre-specified number of algorithm iterations.

Another goal of this paper is to analyze the behavior designed by the CGAFPS, through its use of *rational behavior* [7]. The primary benefit of employing rational behavior is its capacity for optimal search, where optimality is defined as the quickest possible prior information trade off for reduced search effort. A secondary benefit is that rational behavior, when applied to Markov chains (see Section III), is a sufficient condition for responsiveness [4]. To facilitate the analysis, we develop the requisite extension of a SEGS so that the technique in this paper generalizes the CGAFPS. We show that the conditions for the extended technique to achieve rational behavior are highly restrictive, and that there are instances when a SEGS technique finds fit candidates faster than the extended technique.

The implication is that the SEGS scheme in [4], which also employs rational behavior, is more generally applicable for efficient search and responsive behavior design than the extended scheme and the CGAFPS. Hence, the CGAFPS must be modified for use in optimization with dynamic fitness landscapes if rational behavior is desired. However, we also show that the extended technique (and hence the CGAFPS) typically find fit candidates faster than a SEGS; this trade-off is consistent with the No Free Lunch theorem for optimization [8].

B. Background and Technical Approach

Reference [4] proposed an on-line behavior design technique based on the novel concept of *selective generation*,

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which utilizes the ratio of the fitness values of two candidates and a parameter called the level of selectivity. In the limit as the level of selectivity tends to infinity, the scheme guarantees that the selected candidate is a global optimizer. Hence, the SEGS technique in [4] is a generalization of standard optimization. Although rational behavior suggests dynamic transitions that are based on global knowledge, [4] proved that rationality may be achieved through a sequence of dynamic transitions using only local knowledge of the reward function. Thus, a SEGS is also computationally inexpensive at each step.

This paper generalizes selective generation by describing the concept of *multi-selective generation*, which utilizes the fitness value of a candidate optimizer, all fitness values in a neighborhood of the candidate optimizer, and the level of selectivity. That is, multi-selective generation assumes a more global knowledge of the reward function than [4], but the trade-off is faster behavior design.

More specifically, we propose a novel mathematical definition of selection, the *Multi-Select* function, for use in behavior design. We prove that multi-selective generation is a sufficient condition for rational behavior, under certain technical assumptions. Since rational behavior is itself a sufficient condition for responsiveness [4], the resultant multi-selective generation scheme is therefore responsive. We then discuss the limitations imposed by the technical assumptions, and their relevance to the CGAFPS. Lastly, we compare the convergence properties of multi-selective generation with a SEGS.

C. Related Literature

The work in this paper is a Markov Chain Monte Carlo method [9] for optimization. Other probabilistic evolutionary computation approaches include the covariance matrix adaptation evolution strategy [10] and the estimation distribution algorithm [11]. A demonstration of the uniqueness of our work with respect to the applicable literature is available in [4]. Evolutionary computation for dynamic fitness landscapes is a relatively new area of study; for a recent overview, see [12]. The authors of [12] also state that ‘there are no published results that are comparative to the patentable works cited for static environments,’ a failing that this paper seeks to remedy. An example of a modified genetic algorithm for optimization on a dynamic fitness landscape is [13].

The convergence properties of genetic algorithms have been previously analyzed with Markov chains [14], [15]. One novelty of this paper is a study of the use of rational behavior by the CGAFPS, and a convergence comparison of a generalized CGAFPS with the new SEGS technique.

D. Paper Outline

The remainder of the paper is as follows. Section II creates a theoretical framework, defines the *Multi-Select* function, and states the similarity between multi-selective generation and the CGAFPS. Section III presents an abridged version of Markov chain rational behavior. Section IV details some dynamic and convergence properties of multi-selective generation, indicating why the CGAFPS is not typically rational. Section V discusses a few multi-selective generation algorithm characteristics inherited from the SEGS technique. Section VI presents conclusions.

II. Theoretical Foundation

In behavior design, a *cell* is any element of the domain of a reward function, and a *resource* is any input that facilitates a transition between cells. Cells may also be referred to as states or candidate optimizers. A cell utilizes a resource to *reproduce* and generate an offspring, i.e., transition to another cell. The generation process may incorporate features such as mutation, recombination, inheritance, drift, and flow [4]. Consistent with these notions, we make the following definition.

Definition 1: A *generation system* is a triple $\mathscr{E} = (X, R, G)$, where

- X is a *set of n cells*, $X = \{x_1, x_2, \ldots, x_n\}$;
- R is a *set of m resources*, $R = \{r_1, r_2, \ldots, r_m\}$, that can be utilized for cell reproduction;
- $G : X \times R \to X$ is a *generation function* that maps a parent cell and a resource into a descendant cell outcome.

Let $(r_\mu) = (r_1, r_2, \ldots, r_\mu)$ be a sequence of μ resources from R. We define the notation

$$G\left(x, (r_\mu)\right) := G(\ldots G(G(x, r_1), r_2) \ldots, r_\mu) \tag{1}$$

to denote the cell produced by x using sequence (r_μ).

Definition 2: The set of cells, X, of the generation system $\mathscr{E} = (X, R, G)$ is *reachable* through G and R if, for all pairs $(x_1, x_2) \in X^2$, there exists $k \in \mathbb{N}$ and a sequence $(r_k) \in R$ such that $x_2 = G(x_1, (r_k))$.

Note that reachability of the cells of a generation system is identical to that of reachability of the vertices of a directed graph in Graph Theory [16].

In Definition 1, the restriction that the offspring of a cell be itself a cell implies that the set of cells is *closed* [17], since there is no feasible transition to any element outside X. If the set of cells is also reachable, then X is said to be *irreducible* [17].

We associate each cell with a non-zero, positive performance index that is a measure of the fitness of the cell, $F : X \to \mathbb{R}^+$. The notion of fitness facilitates the following novel mathematical definition of selection.

Definition 3: Given a cell set, X, and a fitness function $F : X \to \mathbb{R}^+$, let *Multi-Select* $: X^k \times \mathbb{N} \to X$ be a random function such that if $x_1 \in X, \ldots, x_k \in X$ are any k cells, and $N \in \mathbb{N}$ is the *level of selectivity*, then for all $1 \le i \le k$,

$$\textit{Multi-Select}(x_1, \ldots, x_k, N) = x_i, \text{ with probability}$$
$$\frac{F(x_i)^N}{\sum_{j=1}^{k} F(x_j)^N}. \tag{2}$$

We can now define a multi-selective generation system (MSGS).

Definition 4: A *multi-selective generation system* is a quadruple $\Gamma = (X, R, G, F)$, where

- (X,R,G) is a generation system;
- $F : X \to \mathbb{R}^+$ is a function that evaluates cell fitness;
- the set of cells, X, is reachable through G and R; and
- the dynamics of the system are given by

$$\mathscr{X}(t+1) = \textit{Multi-Select}\,(\mathscr{X}(t), G(\mathscr{X}(t), r_1), \ldots, G(\mathscr{X}(t), r_m), N)\,. \tag{3}$$

In (3), $\mathscr{X}(t)$ denotes the realization of a random cell variable at time t, r_i is a resource where $1 \le i \le m$, $G(\mathscr{X}(t), r_i)$ denotes the offspring of the realized random cell utilizing resource r_i at time t, and $\mathscr{X}(0)$ has a known probability mass function. The cells $G(\mathscr{X}(t), r_i)$, $1 \le i \le m$, constitute the largest neighborhood of $\mathscr{X}(t)$ within which a transition is possible. The fitness values of cells in this neighborhood are required at each step.

Also in (3), the probability of a cell realization at some future time given the present cell realization is conditionally independent of the past time history of cell realizations. Thus, the dynamics of an MSGS form a discrete-time homogeneous Markov chain [9]. This property is useful for the MSGS analysis conducted in Section IV.

The *Multi-Select* function has a number of interesting properties, including:

- For all N and for all $1 \le i \le k$, $1 \le j \le k$,

$$\frac{\Pr[\textit{Multi-Select}(x_1, \ldots, x_k, N) = x_i]}{\Pr[\textit{Multi-Select}(x_1, \ldots, x_k, N) = x_j]} = \left(\frac{F(x_i)}{F(x_j)}\right)^N. \tag{4}$$

  That is, the ratio of the probabilities of selecting any two cells is equal to the ratio of their respective fitnesses raised to the power N.
- For $N = 0$, the values of fitnesses are irrelevant. That is, for all $1 \le i \le k$,

$$\Pr[\textit{Multi-Select}(x_1, \ldots, x_k, 0) = x_i] = 1/k. \tag{5}$$

- When $N \to \infty$, if there is a unique index, I, such that $F(x_i)$ is maximized for $i = I$ then

$$\Pr[\textit{Multi-Select}(x_1, \ldots, x_k, \infty) = x_I] \to 1. \tag{6}$$

- If all the fitnesses are equal then, for all N and for all $1 \le i \le k$,

$$\Pr[\textit{Multi-Select}(x_1, \ldots, x_k, N) = x_i] = 1/k. \tag{7}$$

Reference [4] demonstrates that, for each iteration of the CGAFPS, the ratio of the probability of selecting an unchanged cell as a member of the population for the next generation to the probability of selecting an offspring of this cell (i.e., a mutated and/or recombined version of the cell) as a member of the population for the next generation is proportional to the fitness ratio of this cell and its offspring. If the constant of proportionality is one, then a particular case of (4) is obtained with $N = 1$. For this paper, our extension of the scheme in [4] is such that there is another similarity with the CGAFPS: fitness proportional selection is a particular case of multi-selective generation with $N = 1$.

The concept of multi-selective generation has been previously implemented experimentally with great success. Consider the well-known paper, [18], which describes a system for the evolution of virtual creatures in a fitness landscape that changes frequently because of competition. The work utilizes an *all vs. best* strategy, defined as the competition between all individuals in a generation and a single opponent with the highest fitness from the previous generation. This strategy is what we have called multi-selective generation. The paper states that

> 'the most "interesting" results occurred when the all vs. best competition pattern was used. Both one and two species evolutions produced some intriguing strategies.'

III. Highlights of Markov Chain Rational Behavior

This section presents an overview of the key results of Markov chain rational behavior in [4]. Let $(X, \mathbf{P})$ be a time-homogeneous, irreducible, ergodic Markov chain, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of states of a Markov process, $\mathbf{P} \in \mathbb{R}^{n \times n}$ is the matrix of transition probabilities for these states, and $n < \infty$ is the number of states. Assume that the initial probability distribution over the states is known, i.e., we are given an n-vector $\mathbf{p}(0)$ having elements $p_i(0) = \Pr[\mathscr{X}(0) = x_i]$ for all $x_i \in X$, where $\mathscr{X}(0)$ denotes the state realization at time 0, and we have $\sum_{i=1}^{n} p_i(0) = 1$. Since we have assumed that the states in X are ergodic and irreducible, they admit a unique stationary probability distribution [9], [17]. Let $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_n \end{bmatrix}$ be the row vector of these stationary probabilities, satisfying the constraints $\pi_i > 0\ \forall i$, and $\sum_{i=1}^{n} \pi_i = 1$. Let $F : X \to \mathbb{R}^+$ be a positive fitness function. Let $N \in \mathbb{N}$ be a natural number. We define rational behavior for this Markov chain as follows.

Definition 5: The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is said to *behave rationally* with respect to fitness F with level N if

$$\frac{\pi_i}{\pi_j} = \left(\frac{F(x_i)}{F(x_j)}\right)^N, \; 1 \le i \le n, \; 1 \le j \le n. \tag{8}$$

Each stationary probability can also be explicitly characterized to ensure Markov chain rational behavior, as is indicated by the following theorem.

Theorem 1: The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ behaves rationally with respect to fitness F with level N if and only if

$$\pi_i = \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}, \; 1 \le i \le n. \tag{9}$$

Proof: See [4]. ■

Here, we have a more general, probabilistic version of the optimization of an objective function. A Markov chain that behaves rationally selects the state of maximum fitness with the highest stationary probability, and, in the limit as N approaches ∞, this probability is 1. The problem and solution then revert to one of standard optimization.

Remarkably, rational behavior in Markov chains is the result of a subsidiary optimization.

Theorem 2: The stationary distribution π of the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N solves the optimization problem

$$\min_{\pi_1,\ldots,\pi_n} U(\pi) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \tag{10}$$

subject to the constraints $\sum_{i=1}^{n} \pi_i = 1$, and $\pi_i > 0$, $\forall i$, utilizing the fitness distribution

$$\varphi_i = \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N}, \ 1 \le i \le n. \tag{11}$$

Proof: See [4]. ■

Furthermore, Theorem 2 states that at the optimum, the stationary distribution agrees with the fitness distribution, i.e., $\pi = \varphi$.

Using the notion of entropy, we can interpret (10) as follows. First, we recognize the term $-\ln(\pi_i)$ as the information content of state x_i [19]. Hence, the right hand side of (10) represents the "fitness-expectation of information." Moreover, we have the following.

Corollary 1: The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ behaves rationally with respect to fitness F with level N if and only if its stationary probability distribution minimizes the fitness-expectation of information. At the optimum, this fitness-expectation of information is the entropy of the fitness distribution, i.e.,

$$U^* = H(\varphi) = -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i). \tag{12}$$

Entropy maximization is important for search according to [20]. The relationship between entropy maximization and optimal search is clarified in [21]. Applying the results from [21] and [20], an exponential normalized fitness function relates rational behavior, entropy and optimal search [4]. A scheme with underlying Markov chain dynamics that behave rationally also maximizes the entropy of the fitness distribution when the fitness function is exponential. The implication is that a fitness function like

$$F(x_i) = e^{-\left((z(x_i)-Z)^2\right)} \tag{13}$$

together with a scheme that makes use of rational behavior guarantees "good" behaviors efficiently. In (13), $z : X \to \mathbb{R}$ is an unknown, computable, and possibly changing function that we are interested in. We assume that we are given an element Z in the image of z, and we wish to find $x \in X$ such that $z(x) = Z$, or such that $||z(x) - Z||$ is small.

Reference [4] formally defines responsiveness as the sensitivity of the stationary distribution to changes in fitness. The level of selectivity has an asymptotic effect on responsiveness. Standard optimization ($N \to \infty$) and random optimization ($N = 0$) are non-responsive. Responsiveness is a direct outcome of Markov chain rational behavior, as follows.

Theorem 3: The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is responsive if the chain behaves rationally.

Proof: See [4]. ■

IV. Multi-Selective Generation Systems as Markov Chains that Behave Rationally

This section applies the theory of rational behavior for time-homogeneous, irreducible, ergodic Markov chains (as developed in Section III) to an MSGS as formulated in Section II. We begin with some preliminaries.

Definition 6: Let $\Gamma = (X, R, G, F)$ be a multi-selective generation system. Let $x_i, x_j \in X$ be any two cells. The *descendancy matrix*, δ, has elements

$$\delta_{ij} = \begin{cases} 1 & \text{if } \exists r \in R : x_j = G(x_i, r), \ 1 \le i \le n, \ 1 \le j \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Hence, the descendancy matrix indicates whether it is possible to produce cell x_j in one step from cell x_i, using any resource. Recall that an MSGS follows the stochastic Markov process described by (3). The descendancy matrix is used when specifying a matrix of transition probabilities that describes the cell-to-cell transitions that occur as a result of the multi-selection dynamics. For the MSGS $\Gamma = (X, R, G, F)$, the *matrix of transition probabilities*, $\mathbf{P}$, has elements

$$P_{ij} = \Pr[\mathscr{X}(t+1) = x_j \mid \mathscr{X}(t) = x_i], \tag{15}$$

$$= \Pr[\textit{Multi-Select}(x_i, G(x_i, r_1), \ldots, G(x_i, r_m), N) = x_j \mid \mathscr{X}(t) = x_i] \times \Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i] \tag{16}$$

$$= \begin{cases} \dfrac{F(x_j)^N}{\sum_{k=1}^{m} F(G(x_i, r_k))^N + F(x_i)^N} \delta_{ij}, & \forall j \ne i, \\ 1 - \sum_{\substack{j=1 \\ j \ne i}}^{n} \dfrac{F(x_j)^N}{\sum_{k=1}^{m} F(G(x_i, r_k))^N + F(x_i)^N} \delta_{ij}, & \text{if } j = i. \end{cases} \tag{17}$$

Note that the matrix of transition probabilities in (17) is a stochastic matrix.

A. Dynamic Properties of Multi-Selective Generation Systems

We can now state some dynamic properties of multi-selective generation systems, under certain technical conditions.

Theorem 4: For the ergodic MSGS $\Gamma = (X, R, G, F)$, assume that

i) the descendancy matrix, δ, is symmetric, and
ii) $\forall 1 \le i \le n, \ 1 \le j \le n$ with $\delta_{ij} = 1$,

$$\sum_{k=1}^{m} F(G(x_i, r_k))^N + F(x_i)^N = \sum_{k=1}^{m} F(G(x_j, r_k))^N + F(x_j)^N. \tag{18}$$

Then the Markov chain representing the stochastic dynamics of the ergodic MSGS

1) behaves rationally with fitness F and level N. That is, the row vector $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_n \end{bmatrix}$, where π_i

satisfies (9), is a left eigenvector of $\mathbf{P}$, the matrix of transition probabilities for Γ, with corresponding eigenvalue 1 (i.e., $\pi\mathbf{P} = \pi$). Hence, π is the vector of stationary probabilities for the MSGS.

2) is time-reversible, i.e.,

$$\pi_i P_{ij} = \pi_j P_{ji},\ \forall i,j. \tag{19}$$

Proof: See [4]. ■

As a result of Theorem 3, the stochastic dynamics of the ergodic MSGS with sufficient conditions i) and ii) are responsive.

The symmetry condition i) on the descendancy matrix, δ, implies that there exists a forward and reverse transition between any pair of cells. This condition is similar to the one in [4].

Condition ii) is a restrictive sufficient condition. It states that the sum of the fitness values of possible transitions in the neighborhood of x_i, which includes x_j, is equal to the sum of the fitness values of possible transitions in the neighborhood of x_j, which includes x_i. Although this condition is an extension of one implicitly assumed in [4], which is that the addition of any two fitness values commute, the extended condition ii) may be difficult to satisfy.

If condition ii) is satisfied, then there is a need to evaluate the fitness of cells in a sub-population of candidate optimizers (as in the CGAFPS). For $m > 2$, an ergodic MSGS is more computationally expensive than an ergodic SEGS, but can be less expensive than evaluating the fitness of all elements in the domain of the objective function at the start of the search.

Necessary conditions for rational behavior are as follows.

Theorem 5: For the ergodic MSGS $\Gamma = (X, R, G, F)$, assume that the Markov chain representing the stochastic dynamics of the ergodic MSGS behaves rationally with fitness F and level N. Then

$$\sum_{i=1}^{n} \frac{F(x_i)^N}{\sum\limits_{k=1}^{m} F\left(G\left(x_j, r_k\right)\right)^N + F\left(x_j\right)^N} \delta_{ji} = \sum_{i=1}^{n} \frac{F(x_i)^N}{\sum\limits_{k=1}^{m} F\left(G\left(x_i, r_k\right)\right)^N + F\left(x_i\right)^N} \delta_{ij}. \tag{20}$$

If the Markov chain representing the stochastic dynamics of the ergodic MSGS is also time-reversible, then

$$\frac{\delta_{ji}}{\sum\limits_{k=1}^{m} F\left(G\left(x_j, r_k\right)\right)^N + F\left(x_j\right)^N} = \frac{\delta_{ij}}{\sum\limits_{k=1}^{m} F\left(G\left(x_i, r_k\right)\right)^N + F\left(x_i\right)^N}. \tag{21}$$

Proof: See [4]. ■

For finite N and cell fitness values, (21) is thus a necessary and sufficient condition for rational behavior. Therefore, not satisfying (21) results in behavior that is not rational. Since the CGAFPS is not often applied such that (21) is true with $N = 1$, the CGAFPS does not achieve rational behavior in these instances. Modification is required.

B. Convergence Properties of Multi-Selective Generation Systems

We now compare the convergence rates of an ergodic MSGS and an ergodic SEGS, by utilizing the second largest eigenvalue as a measure of convergence rate — the smaller this value, the more quickly the Markov chain dynamics converge to steady state. We make use of the result in [22] for reversible Markov chains with a common underlying graph. Let the underlying graph representations of the ergodic MSGS and the ergodic SEGS be the same, and let the sufficient conditions for rational behavior with fitness F with level N be satisfied. If $\lambda_2^M < 1$ is the second largest eigenvalue of the MSGS probability transition matrix $\mathbf{P}^M$, and $\lambda_2^S < 1$ is the second largest eigenvalue of the SEGS probability transition matrix $\mathbf{P}^S$ then, from [22],

$$\left(1 - \lambda_2^M\right) \geq \frac{\min\limits_{\{i,j\}\text{ is an edge}} \frac{w_{ij}^M}{w_{ij}^S}}{\max\limits_{i} \frac{\pi_i^M}{\pi_i^S}} \left(1 - \lambda_2^S\right), \tag{22}$$

where $w_{ij}^M = \pi_i^M P_{ij}^M$, $w_{ij}^S = \pi_i^S P_{ij}^S$.

Because both techniques yield rational behavior, $\pi_i^M = \pi_i^S$. Consequently,

$$\left(1 - \lambda_2^M\right) \geq \left(\min_{\{i,j\}\text{ is an edge}} \frac{P_{ij}^M}{P_{ij}^S}\right) \left(1 - \lambda_2^S\right), \tag{23}$$

or

$$\left(1 - \lambda_2^M\right) \geq \alpha \left(1 - \lambda_2^S\right), \tag{24}$$

where, from [4] and (17), $\alpha =$

$$\min_{\substack{\{i,j\} \\ \text{is an edge}}} \frac{F(x_i)^N + F(x_j)^N}{\gamma_{ij}\left(\sum\limits_{\substack{k=1 \\ G(x_i, r_k) \neq x_j}}^{m} F\left(G\left(x_i, r_k\right)\right)^N + F\left(x_i\right)^N + F\left(x_j\right)^N\right)}, \tag{25}$$

where γ_{ij} is a probability value on the interval $(0, 1]$.

In (24), if $\alpha = 1$, then $\lambda_2^M \leq \lambda_2^S$. Since $N \to \infty$ implies that $\alpha \to 1$, non-responsive multi-selective generation converges to steady state faster than a SEGS process.

In (24), if $\alpha > 1$, then $\lambda_2^M \leq \lambda_2^S$. This occurs for a typical application of the SEGS, where the number of resources is large enough so that the probability distribution on these resources (which is indicated by γ_{ij}, and is typically a uniform probability distribution) is small, yet the number of resources is also small enough so that the corresponding MSGS implementation does not select among large fitness neighborhoods at each iteration. For this scenario, a SEGS requires a lot more exploration than the MSGS.

In (24), if $0 < \alpha < 1$, then $\lambda_2^M \leq \alpha\lambda_2^S + 1 - \alpha$. That is, for small α, it is possible that $\lambda_2^S \leq \lambda_2^M$. This occurs when γ_{ij} is large and N is small. The physical interpretation of this scenario is as follows: the resource choice (indicated by γ_{ij}) is biased in such a way that the probability of SEGS transitions to cells of higher fitness is greater than the probability of corresponding MSGS transitions. Hence, the faster SEGS convergence to steady state for this scenario. We believe that such a bias is possible, but is atypical.

The preceding analysis leads to the following conclusion: while the CGAFPS may not satisfy the necessary conditions for rationality and hence not design rational behavior, the algorithm will, in general, converge to steady state faster than a SEGS. Such a trade-off is consistent with the No Free Lunch theorem for optimization [8].

V. Discussion

The MSGS scheme inherits characteristics from the SEGS technique, discussed here. A detailed exposition is in [4].

The level of selectivity N manipulates the trade-off between search *exploration* (search diversity) and search *exploitation* (search intensity in a local search subspace). At very low N, a selective generation algorithm wanders through the search space and may not reach a desired target Z within a user-specified limit of generations. Increases in N cause a corresponding improvement in target tracking. Low N trajectories typically depict excursions away from the desired target; these excursions are minimized as N increases. High N trajectories achieve near perfect target tracking with few excursions. When the target variations are large, lower N trajectories display a more immediate response to the change in target than higher N trajectories. However, the more selective trajectories overtake trajectories with lower N values after a short period of time. An increase in N decreases the number of generations to find a "good" solution.

Initial conditions do play a role in the convergence of a selective generation algorithm. Moreover, a significantly greater average number of generations is required when there are fewer fit solutions in the search space. Since generation systems use resources to discretize the search space, the type of discretization employed by a generation system affects the average number of generations required to find a solution.

The CGAFPS exhibits responsive behavior, which is unsurprising since it is a particular case of a selective generation scheme with $N = 1$. The (1+1) evolutionary strategy behaves like a selective generation scheme with very high N. Note that the MSGS and SEGS algorithms are not related to the cross-entropy method for optimization [23]. Reference [4] provides insight into the connection between responsive optimization and cross-entropy.

The original SEGS technique has been applied to problems in flight mechanics, control of dynamic systems, and artificial intelligence. Similar MSGS versatility is expected.

VI. Conclusions and Future Work

Multi-selective generation extends a viable technique that uses rationality to achieve responsive behavior. Rational behavior is desirable because of its capacity for efficient search. However, the conditions for this extended scheme to behave rationally are highly restrictive. Since the technique is a generalization of the canonical genetic algorithm with fitness proportional selection (CGAFPS), it is unlikely that a typical application of the CGAFPS behaves rationally. Multi-selective generation can find a fit candidate optimizer faster than the original technique that it extends, but exceptions do exist. Therefore, the original scheme in [4] should be preferred for efficient search and responsive behavior design.

Future work includes classifying the types of multi-selective generation systems that satisfy the restrictive conditions for rational behavior. Applications of the technique to finite horizon versions of practical problems, e.g., flapping wing flight as in the original [4], also remain to be explored.

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