Stable Hierarchical Model Predictive Control Using an Inner Loop Reference Model

Chris Vermillion * **Amor Menezes** ** **Ilya Kolmanovsky** ***

* *Altaeros Energies, Cambridge, MA 02140 (e-mail: chris.vermillion@altaerosenergies.com)*
** *Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109 (e-mail: amenezes@umich.edu)*
*** *Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109 (e-mail: ilya@umich.edu)*

Abstract: This paper proposes a novel hierarchical model predictive control (MPC) strategy that guarantees overall system stability. Our method differs significantly from previous approaches to guaranteeing overall stability, which have relied upon a multi-rate framework where the inner loop (low level) is updated at a faster rate than the outer loop (high level), and the inner loop must reach a steady-state within each outer loop time step. In contrast, our approach is aimed at stabilizing the origin of an error system characterized by the difference between the inner loop state and the state specified by a full-order reference model. This makes our method applicable to systems that do not possess the level of time scale separation that is required to apply the multi-rate framework successfully. Stability constraints for the proposed algorithm are derived, and the effectiveness of the proposed reference model-based strategy is shown through simulation on a stirred tank reactor problem, where we demonstrate that the MPC optimization problem remains feasible and the system remains stable and continues to perform well when time scale separation between the inner and outer loops is reduced.

Keywords: Control of constrained systems, Decentralized control, Optimal control theory.

1. INTRODUCTION

This paper focuses on a two-layer inner loop/outer loop hierarchical control structure depicted in Fig. 1, where the ultimate objective is to track a setpoint, r. The actuator and plant represent a cascade wherein a virtual control, v, characterizes an overall force, moment, or generalized effect produced by the actuators at the inner loop, and this virtual control input acts as the single driver to the plant. In the hierarchical control strategy, an outer loop controller sets a desired *virtual control input*, v_{des}, and it is the responsibility of the inner loop to generate control inputs u that drive v close to v_{des}.

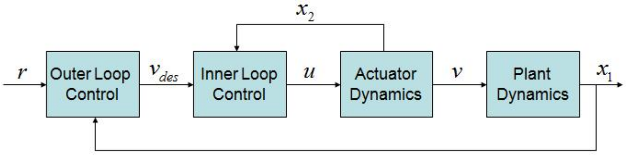

Fig. 1. Basic hierarchical control strategy.

This control approach is employed in a number of automotive, aerospace, and marine applications, such as [Luo (2004)], [Luo (2005)], [Luo (2007)], [Tjonnas (2007)], and [Vermillion (2007)]. The use of MPC for this hierarchical structure has been popularized recently in the works of [Falcone (2008)], [Scattolini (2007)], [Scattolini (2008)], [Scattolini (2009)], and [Picasso (2010)]. Several of these papers have considered the stability problem for hierarchical control for certain classes of systems. In [Scattolini (2007)], [Picasso (2010)], and [Picasso (2010)], a multi-rate hierarchical MPC scheme is proposed, with stability constraints, in which the inner loop is updated at a faster rate than the outer loop, allowing the inner loop several time instances to force v to v_{des}. Specifically, at each outer loop time instant, the steady state corresponding to $v = v_{des}$ is calculated, and the inner loop MPC places a constraint on the optimization that requires the inner loop to reach this steady state within a single outer loop time instant (in addition to several other constraints that make the method work). The strategy proposed in [Scattolini (2007)], [Picasso (2010)], and [Picasso (2010)], represents a very effective way of guaranteeing stability when there is a large time scale separation between the inner and outer loops. However, because the inner loop must reach a steady state within a single outer loop step, the method is impractical for systems without sufficient time scale separation.

The approach proposed in this paper differs from that of [Scattolini (2007)], [Scattolini (2008)], and [Picasso (2010)] in that it attempts to drive the inner loop states to those of a *reference model* rather than to the steady state values corresponding to v_{des}. This reference model is designed

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to represent the desired inner loop dynamics, and the state of the inner loop is guaranteed through our approach to asymptotically track the state of the reference model. Our stability formulation relies on terminal constraint sets for the outer and inner loop, in addition to rate-like constraints that ensure that the optimized MPC trajectories do not vary too much from one instant to the next. We show through the same stirred tank reactor example as was used in [Picasso (2010)] that our algorithm provides for a stable interconnected system under a variety of actuator dynamics, including those that are significantly slower than the ones explored in [Picasso (2010)].

The paper is organized as follows. Section 2 provides the formal problem statement, along with notation that will be used throughout the paper, and Section 3 specifies the details of the control design and summarizes the MPC design parameters in a compact table. Sections 4 and 5 present the theoretical framework behind the construction of the invariant sets and rate-like constraints that are used in our stability formulation. Section 6 presents results relating to successive feasibility, convergence to terminal sets, and, most importantly, overall system stability. Finally, section 7 presents a stirred tank reactor problem that illustrates the effectiveness of our proposed algorithm.

2. PROBLEM STATEMENT

In this paper, we consider two interconnected systems, whose dynamics in discrete time are given by:

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 v(k), \\ x_2(k+1) &= A_2 x_2(k) + B_2 u(k), \\ v(k) &= C x_2(k), \end{aligned} \tag{1}$$

where $v \in \mathbb{R}^q$ represents the virtual control input, $x_1 \in \mathbb{R}^{n_1}$ represents the plant states, which are driven by the virtual control input, v, whereas $x_2 \in \mathbb{R}^{n_2}$ represents the actuator states, which are driven by the real control inputs, $u \in \mathbb{R}^p$, where $p \geq q$. We assume that the pair (A_2, B_2) is controllable from each control input, and, without further loss of generality, that the actuator dynamics of (1) are written in block controllable canonical form (CCF) [Luenberger (1967)].

In this paper, we are interested in stability of the overall system (both inner and outer loops) under constant setpoints, r. Assuming that there exist control inputs that achieve $x_{ss} = r$, we can, without loss of generality, perform a coordinate translation and consider the problem of asymptotic stability with $r = 0$:

Definition 1. (Asymptotic Stability) The origin, $x = 0$, is asymptotically stable if for every $\epsilon > 0$, there exists $\delta_1 > 0$ such that:

$$\|x(0)\|_2 \leq \delta_1 \Rightarrow \|x(k)\|_2 \leq \epsilon, \forall k \geq 0, \tag{2}$$

and there exists $\delta_2 > 0$ such that:

$$\|x(0)\|_2 \leq \delta_2 \Rightarrow lim_{k\to\infty} x(k) = 0. \tag{3}$$

□

Furthermore, the region of attraction, is defined as follows:

Definition 2. (Region of Attraction) The region of attraction of $x = 0$ is the largest set, $R \subset \mathbb{R}^n$, such that if $x(0) \in R$, then $\lim_{k\to\infty} x(k) = 0$. □

3. CONTROL DESIGN FORMULATION

In contrast to other approaches that aim to force v to track v_{des} exactly, our approach relies on the design of an inner loop *reference model*, which describes the ideal input-output behavior from v_{des} to v. This reference model is given by:

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f v_{des}(k), \\ v_{des}^f(k) &= C x_f(k), \end{aligned} \tag{4}$$

where the block diagram description of the overall system under this reference model approach is given in Fig. 2.

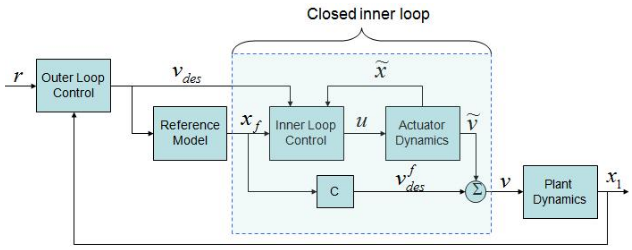

Fig. 2. Hierarchical control strategy under the reference model based approach. This block diagram also conforms to the system structure when the terminal controller is in place in our proposed MPC strategy.

We assume that:

- The reference model is stable, i.e., $re(\bar{\lambda}_i(A_f)) < 0, \forall i$ (we use $\bar{\lambda}$ to distinguish eigenvalues from contraction rates, which are also traditionally denoted by λ);
- The reference model does not share any zeros with unstable poles of A_1 (a mild assumption since the designer has full control over the reference model).

In order to generate an error system between the actuator and reference model states (i.e., $x_2 - x_f$), we require the reference model to be written in block CCF, just as in (1). It follows from this form that the pair (A_f, B_f) is controllable and the reference model can be matched.

The error system between the reference model and actuator states is described by:

$$\begin{aligned} \tilde{x}(k+1) &= A_2\tilde{x}(k) + (A_2 - A_f)x_f(k) + B_2 u(k) \\ &\quad - B_f v_{des}(k), \\ \tilde{v}(k) &= C\tilde{x}(k), \end{aligned} \tag{5}$$

where $\tilde{x}(k) = x_2(k) - x_f(k)$. For notational convenience throughout the paper, because the reference model is embedded in the outer loop, we will introduce the augmented outer loop state, $x_1^{aug} \triangleq \begin{bmatrix} x_1^T & x_f^T \end{bmatrix}^T$, which results in augmented outer loop dynamics given by:

$$x_1^{aug}(k+1) = A_1^{aug} x_1^{aug} + B_1^{aug}\tilde{v}(k) + B_f^{aug} v_{des}(k) \tag{6}$$

where:

$$\begin{aligned} A_1^{aug} &= \begin{bmatrix} A_1 & B_1 C \\ 0 & A_f \end{bmatrix}, \\ B_1^{aug} &= \begin{bmatrix} B_1^T & 0 \end{bmatrix}^T, \\ B_f^{aug} &= \begin{bmatrix} 0 & B_f^T \end{bmatrix}^T. \end{aligned} \tag{7}$$

The overall framework with MPC is given by Fig. 3, where an MPC optimization is carried out whenever the outer *or* inner loop states are outside of predetermined λ-contractive terminal constraint sets, G_1 and G_2 respectively, but a closed-form terminal control law is active once the inner *and* outer loop states have reached the terminal sets. The mathematical description of the outer

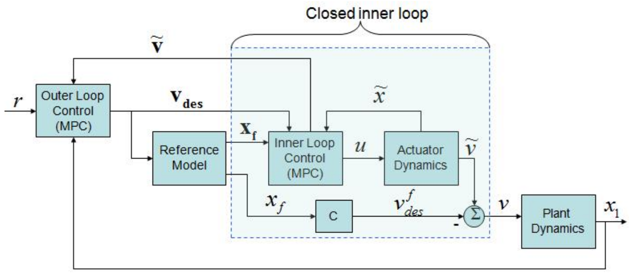

Fig. 3. Hierarchical control strategy when model predictive control is active.

loop control law is:

$$v_{des}(k) = \begin{cases} \mathbf{v^o_{des}}(k|k) \quad if \ x_1^{aug}(k) \in G_1 \ and \ \tilde{x}(k) \in G_2, \\ -K_1 x_1^{aug}(k) \quad , \quad otherwise \end{cases}$$

Here, K_1 is the terminal control gain and $\mathbf{v}^o_{des}(k)$ is the optimized control input sequence from the outer loop MPC optimization, given by:

$$\mathbf{v^o_{des}}(k) = \arg\min J_1(\mathbf{v_{des}}(k)|x_1^{aug}(k), \tilde{\mathbf{v}}(k)), \qquad (8)$$

subject to the constraints:

$$\begin{aligned} \mathbf{x_1^{aug}}(k+N-1|k) &\in G_1, \\ \mathbf{x_1^{aug}}(k+N|k) &\in \lambda_1 G_1, \qquad (9) \\ \|\mathbf{v_{des}}(k+i|k) - \mathbf{v_{des}}(k+i|k-1)\| &\leq (\delta^{max}_{v_{des}})\beta^k, \\ i &= 0 \dots N-2, \end{aligned}$$

and cost function:

$$J_1(\mathbf{v_{des}}(k)|x_1^{aug}(k), \tilde{\mathbf{v}}(k)) = \sum_{i=k}^{k+N-1} g_1(\mathbf{x_1^{aug}}(i|k), \mathbf{v_{des}}(i|k)). \qquad (10)$$

Note that the **bold** versions of variables represent their corresponding N-step MPC sequences (over the N-step horizon), i.e.,

$$\begin{aligned} \mathbf{v_{des}}(k) &= [\, \mathbf{v_{des}}(k|k) \ \dots \ \mathbf{v_{des}}(k+N-1|k) \,], \\ \mathbf{x_1^{aug}}(k) &= [\, \mathbf{x_1^{aug}}(k|k) \ \dots \ \mathbf{x_1^{aug}}(k+N|k) \,]. \qquad (11) \end{aligned}$$

The mathematical description of the inner loop control law is:

$$u(k) = \begin{cases} \mathbf{u^o}(k|k) \ if \ x_1^{aug}(k) \in G_1 \ and \ \tilde{x}(k) \in G_2 \\ u_t(k) \quad , \quad otherwise \end{cases}$$

where $u_t(k) = K_{21}v_{des}(k) + K_{22}x_f(k) - K_{23}\tilde{x}(k)$. Here, $\mathbf{u^o}(k)$ is the optimized control input sequence from the outer loop MPC optimization, given by:

$$\mathbf{u^o}(k) = \arg\min J_2(\mathbf{u}(k)|\tilde{x}(k), \mathbf{x_f}(k)), \qquad (12)$$

subject to the constraints:

Table 1. MPC Design Parameters for Stability Constraints

Parameter	Description
G_1	outer loop terminal constraint set
G_2	inner loop terminal constraint set
K_1	outer terminal control gain matrix
$K_{21,22,23}$	inner terminal control gain matrices
λ_1	outer contraction rate (< 1)
λ_2	inner contraction rate (< 1)
$\delta^{max}_{v_{des}}$	rate-like constraint on outer loop MPC
δ^{max}_u	rate-like constraint on inner loop MPC
β	any scalar that is < 1

$$\begin{aligned} \tilde{\mathbf{x}}(k+N|k) &\in \lambda_2 G_2, \\ \|\mathbf{u}(k+i|k) - \mathbf{u}(k+i|k-1)\| &\leq (\delta^{max}_u)\beta^k, \qquad (13) \\ i &= 0 \dots N-2 \\ \mathbf{u}(k+i|k) &\in U, i = 0 \dots N-1, \end{aligned}$$

where U classifies the actuator saturation limits of u, and cost function structure:

$$J_2(\mathbf{u}(k)|\tilde{x}(k), \mathbf{x_f}(k)) = \sum_{i=k}^{k+N-1} g_2(\tilde{\mathbf{x}}(i|k), \mathbf{u}(i|k)). \qquad (14)$$

The subsequent sections will describe how the terminal constraint sets, G_1 and G_2, are constructed, as well as how the rate-like constraints, specified by $\delta^{max}_{v_{des}}$ and δ^{max}_u are determined once contraction coefficients λ_1, λ_2 are chosen. For convenience, all of the design parameters of the MPC optimization that are critical for applying the stability constraints are summarized in Table 1.

4. DERIVING λ-CONTRACTIVE TERMINAL CONSTRAINT SETS AND CONTROL LAWS

In this section, we will first derive control laws that, in the absence of constraints, will lead to overall system stability. Having derived these control laws, we will then show that there exist λ- contractive sets G_1 and G_2, such that once x_1^{aug} and $\tilde{x}$ enter $G_{1,2}$, they remain there (and in fact are driven further into the sets at the next instant).

Proposition 3. (Terminal control laws) There exist control laws $v_{des}(k) = -K_1 x_1^{aug}(k)$ and $u(k) = K_{21}v_{des}(k) + K_{22}x_f(k) - K_{23}\tilde{x}(k)$ which, when substituted into the system dynamics, yield:

$$\begin{aligned} x_1(k+1) &= (A_1^{aug} - B_1^{aug}K_1)x_1^{aug}(k) + B_1^{aug}\tilde{v}(k), (15) \\ \tilde{x}(k+1) &= A_f \tilde{x}(k), \end{aligned}$$

where $re(\bar{\lambda}_i(A_1^{aug} - B_1^{aug}K_1)) < 0, \forall i$. $\square$

The existence of the outer loop controller follows from the stabilizability of (A_1^{aug}, B_1^{aug}), and the existence of the inner loop controller is easily shown when both the actuator and reference model dynamics are cast in CCF.

Now, we show that λ-contractive sets, conforming to the definition in [Lin (2004)], G_1 and G_2 exist for the outer and inner loops, respectively.

Proposition 4. (Existence of λ-contractive sets) There exist sets $G_1 \in \mathbb{R}^{n_1+n_2}$ and $G_2 \in \mathbb{R}^{n_2}$, along with scalars $\lambda_1 < 1$ and $\lambda_2 < 1$ such that if:

$$
\begin{aligned}
x_1^{aug}(k) &\in G_1, \\
\tilde{x}(k) &\in G_2, \\
v_{des}(k) &= -K_1 x_1(k), \\
u(k) &= K_{21} v_{des}(k) + K_{22} x_f(k) - K_{23}\tilde{x}(k),
\end{aligned} \tag{16}
$$

then:

$$
\begin{aligned}
u(k) &\in U, \\
x_1^{aug}(k+1) &\in \lambda_1 G_1, \\
\tilde{x}(k+1) &\in \lambda_2 G_2.
\end{aligned} \tag{17}
$$

□

The proof follows from the construction of outer and inner loop Lyapunov functions, $V_1(x_1^{aug})$ and $V_2(\tilde{x})$ and bounds $V_{1,2}^*$ such that, under the terminal control laws, when $V(x_1^{aug}(k)) < V_1^*$ and $V(\tilde{x}(k)) < V_2^*$, the disturbance presented to the inner/outer loop from the other loop is small enough that:

(1) $V_1(x_1^{aug}(k+1)) < \lambda_1 V_1^*$,
(2) $V_2(\tilde{x}(k+1)) < \lambda_2 V_2^*$.

The proof of Prop. 4 provides the method by which one can construct G_1 and G_2, and determine suitable values for λ_1 and λ_2, by means of designing appropriate Lyapunov functions $V_1(x_1^{aug})$ and $V_2(\tilde{x})$, respectively.

5. DERIVING RATE-LIKE CONSTRAINTS ON CONTROL INPUTS AND DESIRED VIRTUAL CONTROL INPUTS

Having discussed the existence and construction of the terminal control laws (with gains K_1, $K_{21,22,23}$), terminal constraint sets ($G_{1,2}$), and contraction rates ($\lambda_{1,2}$) in Section 4, we now turn to the construction of the rate-like constraints, $\delta_{v_{des}}^{max}$ and δ_u^{max}, which limit the variation of $\mathbf{v_{des}}$ and $\mathbf{u}$ trajectories from one time instant to the next.

We begin with the following proposition, which follows from examination of the time series representation of the x_1^{aug} trajectory:

Proposition 5. (Robustness of outer loop MPC to variation in $\tilde{\mathbf{v}}$) - Suppose that, given

$$\tilde{\mathbf{v}}(\mathbf{k}) = [\, \tilde{\mathbf{v}}(k|k) \, \dots \, \tilde{\mathbf{v}}(k+N-1|k) \,],$$

there exists a trajectory

$$\mathbf{v_{des}}(k) = [\, \mathbf{v_{des}}(k|k) \, \dots \, \mathbf{v_{des}}(k+N-1|k) \,],$$

that yields $\mathbf{x_1^{aug}}(k+N|k) \in \lambda_1 G_1$. Then it is possible to choose $\epsilon_{\tilde{v}}^{max} > 0$ such that if $\|\tilde{\mathbf{v}}(k+i|k) - \tilde{\mathbf{v}}(k+i|k+1)\| \leq \epsilon_{\tilde{v}^{max}}, i = 1 \dots N-1$ and $\mathbf{v_{des}}(k+i|k+1) = \mathbf{v_{des}}(k+i|k), i = 1 \dots N-1$, then $\mathbf{x_1^{aug}}(k+N|k+1) \in G_1$. □

We arrive at a very similar conclusion regarding the robustness of the inner loop MPC to variation in $\mathbf{x_f}$:

Proposition 6. (Robustness of inner loop MPC to variation in $\mathbf{x_f}$) - Suppose that, given

$$\mathbf{x_f}(\mathbf{k}) = [\, \mathbf{x_f}(k|k) \, \dots \, \mathbf{x_f}(k+N|k) \,],$$

there exists a trajectory

$$\mathbf{u}(k) = [\, \mathbf{u}(k|k) \, \dots \, \mathbf{u}(k+N-1|k) \,],$$

that yields $\tilde{\mathbf{x}}(k+N|k) \in \lambda_2 G_2$. Then it is possible to choose $\epsilon_{x_f}^{max} > 0$ such that if $\|\mathbf{x_f}(k+N|k) - \mathbf{x_f}(k+N|k+1)\| \leq \epsilon_{x_f}^{max}$ and $\mathbf{u}(k+i|k+1) = \mathbf{u}(k+i|k), i = 1 \dots N-1$, then $\tilde{\mathbf{x}}(k+N|k+1) \in G_2$. □

The proof of Prop. 6 is straightforward, given that x_f represents an output disturbance to $\tilde{x}$, and therefore any restriction on the variation in x_f will map directly to a restriction on the variation in $\tilde{x}$.

It is possible to convert the state constraints of Props. 5 and 6 to input constraints (on $\mathbf{v_{des}}$ and $\mathbf{u}$), which are easily enforced and will always result in a feasible optimization problem. These input constraints are given in the following propositions:

Proposition 7. (Converting constraints on $\tilde{\mathbf{v}}$ to constraints on $\mathbf{u}$) - Suppose that $\|C\|\epsilon_{x_f}^{max}\epsilon_{\tilde{v}}^{max}$ and that the input trajectory

$$\mathbf{u}(k-1) = [\, \mathbf{u}(k-1|k-1) \, \dots \, \mathbf{u}(k+N-2|k-1) \,],$$

in conjunction with the external input trajectory

$$\mathbf{x_f}(k-1) = [\, \mathbf{x_f}(k-1|k-1) \, \dots \, \mathbf{x_f}(k+N-1|k-1) \,],$$

generates the trajectory

$$\tilde{\mathbf{v}}(k) = [\, \tilde{\mathbf{v}}(k|k) \, \dots \, \tilde{\mathbf{v}}(k+N-1|k) \,].$$

Then there exists $\delta_u^{max} > 0$ such that if $\|\mathbf{u}(k+i|k) - \mathbf{u}(k+i|k-1)\| \leq \delta_u^{max}$, $i = 0 \dots N-2$, then $\|\tilde{\mathbf{v}}(k+i|k+1) - \tilde{\mathbf{v}}(k+i|k)\| \leq \epsilon_{\tilde{v}}^{max}$, $i = 0 \dots N-1$. □

The requirement that $\|C\|\epsilon_{x_f}^{max} \leq \epsilon_{\tilde{v}}^{max}$ arises from the fact that x_f acts as an external disturbance to $\tilde{x}$, and if x_f is allowed to vary too much, relative to $\tilde{v}$, then it is impossible to guarantee that the variation in $\tilde{v}$ will be sufficiently small, regardless of how small one takes δ_u^{max}. No such imposition exists for the conversion of constraints on $\mathbf{x_f}$ to constraints on $\mathbf{v_{des}}$, however, which is presented in the following proposition:

Proposition 8. (Converting constraints on $\mathbf{x_f}$ to constraints on $\mathbf{v_{des}}$) - Suppose that the trajectory

$$\mathbf{v_{des}}(k) = [\, \mathbf{v_{des}}(k|k) \, \dots \, \mathbf{v_{des}}(k+N-1|k) \,],$$

in conjunction with the external input trajectory

$$\tilde{\mathbf{v}}(k) = [\, \tilde{\mathbf{v}}(k|k) \, \dots \, \tilde{\mathbf{v}}(k+N-1|k) \,],$$

generates the trajectory

$$\mathbf{x_f}(k) = [\, \mathbf{x_f}(k|k) \, \dots \, \mathbf{x_f}(k+N|k) \,].$$

Then there exists $\delta_{v_{des}}^{max} > 0$ such that if $\|\mathbf{v_{des}}(k+i|k+1) - \mathbf{v_{des}}(k+i|k)\| \leq \delta_{v_{des}}^{max}$, $i = 0 \dots N-1$, then $\|\mathbf{x_f}(k+i|k+1) - \mathbf{x_f}(k+i|k)\| \leq \epsilon_{x_f}^{max}$, $i = 0 \dots N$. □

6. SUCCESSIVE FEASIBILITY, CONVERGENCE, AND STABILITY

In this section, we bring together the constraints derived in sections 4 and 5. We show how these constraints result in successive feasibility of the MPC optimization problem and asymptotic stability of the overall system, with a region of attraction that is identical to the set of states for which the initial optimization problem is feasible.

6.1 Successive Feasibility

Because the rate-like constraints cannot be applied at step $k = 0$ (since there is no step $k = -1$ against which to compare), we make the following initial feasibility assumption for step $k = 0$:

Initial Feasibility Assumption - There exists a set $X \in \mathbb{R}^{n_1+2n_2}$, such that if $\left[\, x_1^{aug}(0)^T \ \tilde{x}(0)^T \,\right]^T \in X$, then $\mathbf{v_{des}}(0)$ and $\mathbf{u}(0)$ can be chosen and are chosen such that $|\tilde{\mathbf{v}}(i|1) - \tilde{\mathbf{v}}(i|0)| \leq \epsilon_{\tilde{v}}$ and the MPC optimization problem is feasible.

Given this assumption, we are now ready to state the successive feasibility result.

Proposition 9. (Successive feasibility) Suppose that the initial conditions satisfy $\left[\, x_1^{aug}(0)^T \ \tilde{x}(0)^T \,\right]^T \in X$. Then both the outer and inner loop MPC optimizations are feasible at every step, $k \geq 0$. □

The proof follows from the rate-like constraints imposed on $\mathbf{v_{des}}(k)$ and $\mathbf{u}(k)$. Specifically, if the variation in $\mathbf{v_{des}}$ and $\mathbf{u}$ is sufficiently small from step k to $k+1$, then the optimization problem remains feasible at step $k+1$.

6.2 Convergence

Having shown that the optimization problem is successively feasible, the next step is to show that the control law does in fact result in convergence to $G_{1,2}$. This is given in the following proposition:

Proposition 10. (Convergence to $G_{1,2}$) Suppose that the initial conditions satisfy $\left[\, x_1^{aug}(0)^T \ \tilde{x}(0)^T \,\right] \in X$. Then there exists a scalar integer $N^* > 0$ such that, after applying the MPC algorithm for N^* steps, we have $x_1^{aug}(N^*) \in G_1$ and $\tilde{x}(N^*) \in G_2$. □

The proof relies on the fact that the variation in $\mathbf{v_{des}}$ and $\mathbf{u}$ is not only limited, but is also required to decay over time (through the use of $\beta < 1$ in (9) and (13)).

6.3 Overall Stability

We are now poised to state our main result, namely asymptotic stability of the origin of the overall system, with region of attraction X:

Proposition 11. (Asymptotic stability) Under the MPC controller, the origin, $x_1^{aug} = 0$, $\tilde{x} = 0$, is asymptotically stable with region of attraction X. □

The proof contains two parts. First, local asymptotic stability with region of attraction $\{(x_1^{aug}, \tilde{x}) : x_1^{aug} \in G_1, \tilde{x} \in G_2\}$ is shown by demonstrating that both the inner and outer loop systems are input-to-state stable (ISS) and the small gain condition is satisfied within this (invariant) region of attraction. The small gain condition is particularly easy to verify, since the asymptotic tracking of the inner loop reference model results in an inner loop l_2 gain of 0 from v_{des} and x_f to $\tilde{v}$; thus, any finite outer loop input-output gain satisfies the small gain condition. Through the use of MPC, the region of attraction is enlarged to X.

7. APPLICATION EXAMPLE - STIRRED-TANK REACTOR

We now turn to a stirred-tank reactor system first described in [Mhaskar (2006)] and used by [Picasso (2010)] to illustrate the effectiveness of multi-rate hierarchical MPC. The system is designed to control the reactor temperature (T) as well as the concentrations of two species (C_A and C_B). This is done using auxiliary heating from a fluid with a rate of heat input (or removal) Q_{aux}, an inlet steam (which also provides heating) with temperature T_{A0}, and reactant concentration C_{A0}. Finally, Q_{aux}, T_{A0}, and C_{A0} are manipulated through actuators u_1, u_2, and u_3. Sampling the system at a 0.01 minute time step, we arrive at the following outer loop system dynamics:

Table 2. Actuator Dynamic Characteristics

Characteristic	ζ	ω_1	ω_2	ω_3
Nominal	0.2	150	300	500
Slower	0.1	150	300	500
Slowest	0.04	150	300	500

$$x_1(k+1) = \begin{bmatrix} 1.134 & 1.279 & 0 \\ -0.0008454 & 0.9453 & 0 \\ 0.0008454 & 0.005907 & 0.9512 \end{bmatrix} x_1(k) + \begin{bmatrix} 0.01066 & 0.03157 \\ 0 & 0.04861 \\ 0 & 0.0001458 \end{bmatrix} v(k)$$

where:

$$x_1 = [\, \delta T \ \delta C_A \ \delta C_B \,]^T, \qquad (18)$$
$$v = [\, 0.0043\delta Q_{aux} + 4.998\delta T_{A0} \ \delta C_{A0} \,]^T.$$

where δ denotes deviation from nominal values (which result in the system being controlled to a desired steady-state setpoint). This particular linear model has been generated around the setpoint $T = 359$ K, $C_A = 3.59 \frac{kmol}{kg}$, and $C_B = 0.41 \frac{kmol}{kg}$. Because T_{aux} and T_{A0} enter the system at the same point, their effects can be combined into a single virtual control input.

We will consider second order dynamics expressed in continuous time as:

$$V_1(s) = \frac{748\omega_1^2}{s^2 + \zeta\omega_1 s + \omega_1^2} U_1(s) + \frac{100\omega_2^2}{s^2 + \zeta\omega_2 s + \omega_2^2} U_2(s)$$
$$V_2(s) = \frac{4\omega_3^2}{s^2 + \zeta\omega_3 s + \omega_3^2} U_3(s), \qquad (19)$$

for a variety of actuator parameters, summarized in Table 2. In order to reduce conservatism in the calculation of invariant sets and rate-like constraints, raw actuator commands have been scaled by their saturation limits, such that actuator saturation constraints around the nominal setting are given by:

$$-1 \leq u_{1,2,3} \leq 1. \qquad (20)$$

Because the discrete-time actuator dynamics have relative degree 1, and all of their zeros are stable, we consider reference models of the *minimal* form:

$$F_{1,2}(z) = \frac{1-a}{z-a}, \qquad (21)$$

where the reference model is made full order through stable pole-zero cancelations. The outer loop controller is designed through simple pole placement, and the inner loop controller is designed through solving the reference model matching equations. Having designed both controllers, the boundaries of the constraint sets, G_1 and G_2 are selected as level sets of outer and inner loop quadratic Lyapunov functions:

Table 3. Reference Model and MPC Design Parameters for the Application Example

Actuator	a	λ_1	λ_2	δ_u^{max}	$\delta_{v_{des}}^{max}$
Nominal	0.368	0.941	0.837	0.955	0.21
Slower	0.607	0.941	0.857	0.955	0.182
Slowest	0.819	0.941	0.915	0.955	0.135

$$V_1(x_1^{aug}) = (x_1^{aug})^T Q x_1^{aug}, \qquad (22)$$
$$V_2(\tilde{x}) = \tilde{x}^T P \tilde{x},$$

where Q and P are solutions of the discrete-time Lyapunov equations for the closed outer and inner loops, respectively.

Table 3 provide the values for $\lambda_{1,2}$, δ_u^{max}, and $\delta_{v_{des}}^{max}$ for each actuator characterization and corresponding choice of reference model and controller, taking the MPC horizon length to be $N = 20$. Here, $\lambda_2^* \triangleq 1 - \alpha_2$, where α_2 is defined as in Prop. 4. With slower actuator dynamics, the contraction rates within the inner loops are closer to 1 (slower) and, consequently, less variation in $\mathbf{v_{des}}$ is allowable from one optimization to the next.

For our simulation results, we consider the use of quadratic cost functions for both the inner and outer loop MPC, defined as follows:

$$g_1(x_1^{aug}, v_{des}) = (x_1^{aug})^T Q_1 x_1^{aug} + v_{des}^T R_1 v_{des},$$
$$g_2(\tilde{x}, u) = \tilde{x}^T Q_2 \tilde{x} + u^T R_2 u, \qquad (23)$$

subject to the stability constraints outlined in (9) and (13). We consider the problem of stabilizing the overall system to a setpoint of $T = 359$ K, $C_A = 3.59 \ \frac{kmol}{kg}$, and $C_B = 0.41 \ \frac{kmol}{kg}$. The resulting time trajectories are shown in Fig. 4, which demonstrates that our MPC formulation results in a stable closed loop system in the case of each actuator characteristic.

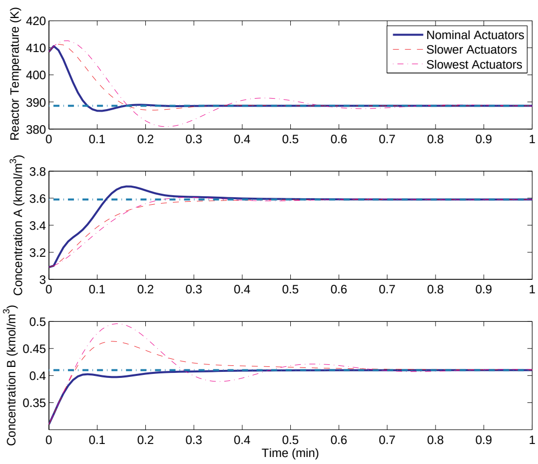

Fig. 4. Simulation results for the stirred-tank reactor example.

8. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed a novel alternative approach to hierarchical MPC that relies on an inner loop reference model rather than a multi-rate approach for achieving overall system stability. This new approach broadens the class of systems for which overall stability of a hierarchical MPC framework can be guaranteed. Future work will examine inexact reference model matching, retroactivity of plant states, and non-constant setpoints.

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