Efficient and Responsive Stochastic Optimization

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Abstract: This paper is devoted to the problem of behavior design, which generalizes the standard global optimization problem of finding an optimizer by producing, on the search space, a probability density function referred to as the behavior. The generalization depends on a parameter, the level of selectivity, such that as this parameter tends to infinity, the behavior becomes a delta function at the location of the global optimizer. The motivation for this generalization is that traditional off-line global optimization is unresponsive to perturbations of the objective function. A novel approach to inexpensive responsiveness is to utilize the theory of Selective Evolutionary Generation Systems (SEGS), which sequentially and probabilistically selects a candidate optimizer based on the ratio of the fitness values of two candidates and the level of selectivity. Using time-homogeneous, irreducible, ergodic Markov chains to model a sequence of local, and hence inexpensive, dynamic transitions, this paper shows that such transitions result in "rational" behavior, and the efficient and responsive search for an optimizer.

Keywords: Monte Carlo methods; adaptation and learning in physical agents; stochastic optimization; evolutionary algorithms; self-reproducing systems.

1. INTRODUCTION

1.1 Motivation and Goals

This paper presents the theory and application of a novel approach to efficient, on-line, inexpensive, responsive stochastic optimization and search. More specifically, this work is concerned with the efficient optimization of an unknown objective function by finding a probability distribution on the search space (for instance, a delta function at the location that optimizes the objective function, i.e., off-line optimization) using a scheme that is responsive to perturbations of the objective function. Such a scheme is motivated by the non-responsiveness of off-line optimization techniques (Dennis and Schnabel, 1996; Ortega and Rheinboldt, 2000; Luenberger, 2003; Boyd and Vandenberghe, 2004) to perturbations of the objective function. Small changes in the objective function may change the sought probability distribution when the optimizer depends continuously or discontinuously on the perturbation, thereby requiring computationally expensive repetitions of off-line optimization. In practice, the objective function on which a candidate optimizer is implemented may be different from that for which the candidate optimizer was determined.

Hence, the goal of this paper is to develop a way of optimally searching for a desirable outcome, such that desirable outcomes are found even if outcome desirability changes. The theoretical solution to this kind of optimization is illustrated in Menezes (2010).

An evolutionary computation approach is utilized in this work. Evolutionary computation for dynamic fitness landscapes is a relatively new and uncharted area of study (for a recent overview, see Dempsey et al. (2009)). As demonstrated in Menezes (2010), the proposed approach is different from alternative optimization algorithms like reinforcement learning (Sutton and Barto, 1998), simulated annealing (Kirkpatrick et al., 1983), genetic algorithms (Goldberg, 1989; Davis, 1991; Mitchell, 1996), and evolutionary strategies (Rechenberg, 1971; Schwefel, 1995; Beyer and Schwefel, 2002). The approach is also different from on-line optimization methods (Ascheuer et al., 1999; Albers, 2003; Hentenryck and Bent, 2006), which are not currently designed for responsiveness. The approach can be shown to be less computationally expensive than the sequential repetition of off-line optimization techniques. Although cross-entropy does appear in this theory, the algorithm here is not related to the cross-entropy method for optimization (Rubinstein and Kroese, 2004). Moreover, this work provides insight into the connection between responsive optimization and cross-entropy. The approach is formulated as a stochastic process that employs *rational behavior* (Meerkov, 1979), and is related to a Markov chain Monte Carlo method.

1.2 Problem Definition

Let X be a search space. In the context of evolutionary computation, X is the set of genotypes. The problem of *behavior design* seeks 1) a probability density function (referred to as the behavior) $\phi_X : X \to \mathbb{R}^+$ that accomplishes specified objectives described below, and 2) dynamic transition laws that cause the variable x to be distributed according to ϕ_X, i.e., to exhibit the behavior specified by ϕ_X.

Let $z : X \to Z$ be an unknown, computable, and possibly changing function that we are interested in. The set Z is a metric space, the set of phenotypes. Suppose that we are given a desired element z_{des} in the image of z, and we wish to find $x \in X$ such that $||z(x) - z_{des}||$ is small. Formally, we want to design a behavior ϕ_X that achieves a known expected value Y, i.e.,

$$\mathrm{E}_{\phi_X}[||z(x) - z_{des}||] = Y, \tag{1}$$

and we refer to this expectation as *goodness*. In the above, Y is effectively a tolerance on what is considered to be good behavior, i.e., it is a limitation on the variance of the behavior. Let $y(x) = ||z(x) - z_{des}||$.

We also desire the behavior ϕ_X to be *responsive* to perturbations in z, i.e.,

$$\frac{\partial \phi_X}{\partial z} \neq 0. \tag{2}$$

The scheme to find ϕ_X should be *efficient* in that it trades off prior information about X for search effort savings as quickly as possible.

Let $f : Z \to \mathbb{R}^+$. We allow the behavior design method to employ a function $F : X \to \mathbb{R}^+ : x \mapsto F(x) = (f \circ z)(x) = f(z(x))$, a real-valued, positive fitness function. In the theoretical discussion that follows, we keep F arbitrary to maintain generality; however, we would also like to determine if efficient behavior design specifies F.

1.3 Original Contributions

The original contributions of this work include:

- A novel mathematical definition of selection, the *Select* function, for use in behavior design.
- A proposition that selective generation is a sufficient condition for rational behavior.
- An analysis demonstrating that rational behavior can lead to optimal search.
- A novel mathematical definition of responsiveness called *resilience*.
- A proposition that rational behavior is a sufficient condition for resilience.
- An analysis of the effect that the level of selectivity has on resilience.

1.4 Paper Outline

The remainder of the paper is as follows. Section 2 highlights the applicable literature. Section 3 presents the fundamentals of a novel scheme for fitness-based selection. Section 4 proves that a sufficient condition for resilience is rational behavior, explains why rational behavior is desirable, demonstrates that resilience may be achieved inexpensively at each step of the scheme, and discusses the relationship with Markov chain Monte Carlo methods. Section 5 presents conclusions.

2. RELEVANT LITERATURE

2.1 Self-X Systems

Self-X systems are systems that are capable of self-assembly, self-organization, self-repair, self-reconfiguration, self-replication, or self-reproduction. The origins of this paper stem from the study of *self-reproducing systems*, a field inspired by the work of John von Neumann (1966). A comprehensive overview of self-replication is documented in Sipper (1998) and Freitas and Merkle (2004).

2.2 Resilience

The concept of resilience was introduced in the seminal work of Holling (1973), and a survey of the many definitions of resilience is available in Brand and Jax (2007). This paper adopts a general notion of resilience: a system is considered to be resilient if it exhibits a response to a disturbance. As long as such a response exists, the characteristic nature of the systems under consideration will ensure that either recovery from the disturbance to the original steady state equilibrium takes place, or a transition to a new optimal equilibrium occurs. Each of these two features is a traditional meaning of resilience.

2.3 Rational Behavior

This paper utilizes the general theory of rational behavior developed in Meerkov (1979). A dynamic system with a decision space is *rational* if each trajectory of this system in the space is

(1) *ergodic*: the trajectory explores all decisions in the decision space, and
(2) *selective*: the trajectory slows down in the vicinity of the most advantageous decisions, i.e., the ratio of the mean time of stay of the trajectory in the vicinity of a more favorable decision to the mean time of stay of the trajectory in the vicinity of a less favorable decision is larger than unity.

Hence, the theory suggests the possibility of rapid convergence to the optimal state of a dynamic system. Unfortunately, global system knowledge may be required to determine how advantageous a state is.

The hypothesis that the theory of Meerkov (1979) yields additional benefits when suitably employed for optimization is validated in this paper. In addition, it is shown that local knowledge of the objective function is sufficient to guarantee rationality.

3. THEORETICAL FOUNDATIONS OF SELECTIVE EVOLUTIONARY GENERATION SYSTEMS

3.1 Theory of Selective Evolutionary Generation Systems

In behavior design, a *cell* is any element of the domain of a reward function, and a *resource* is any input that facilitates a transition between cells. Cells may also be referred to as states or candidate optimizers. A cell utilizes a resource to *reproduce* and generate an offspring, i.e., transition to another cell. Furthermore, it is possible that resources are chosen probabilistically. Consistent with these notions, SEGS theory makes the following definition.

Definition 1. An *evolutionary generation system* is a quadruple $\mathcal{E} = (X, R, P, G)$, where

- X is a *set of* n *cells*, $X = \{x_1, x_2, \ldots, x_n\}$;
- R is a *set of* m *resources*, $R = \{r_1, r_2, \ldots, r_m\}$, that can be utilized for cell reproduction;
- $P : R \to (0, 1]$ is a *probability mass function on* R, given by $P(r_i) = \Pr[\mathcal{R} = r_i] = p_i$, $\sum_{k=1}^{m} p_k = 1$; and
- $G : X \times R \to X$ is a *generation function* that maps a parent cell and a resource into a descendant cell outcome.

Note that for each resource $r \in R$, we assume that an inexhaustible supply is available.

Let $(r_\mu) = (r_1, r_2, \ldots, r_\mu)$ be a sequence of μ resources from R. We define the notation

$$G(x, (r_\mu)) := G(\ldots G(G(x, r_1), r_2) \ldots, r_\mu) \quad (3)$$

to denote the cell produced by x using sequence (r_μ).

Definition 2. The set of cells, X, of the evolutionary generation system $\mathcal{E} = (X, R, P, G)$ is *reachable* through G and R if, for all pairs $(x_1, x_2) \in X^2$, there exists $k \in \mathbb{N}$ and a sequence $(r_k) \in R$ such that $x_2 = G(x_1, (r_k))$.

Note that reachability of the cells of an evolutionary generation system is identical to that of reachability of the vertices of a directed graph in Graph Theory (Diestel, 2005).

In Definition 1, the restriction that the offspring of a cell be itself a cell implies that the set of cells is *closed* (Cassandras and Lafortune, 2008), since there is no feasible transition to any element outside X. If the set of cells is also reachable, then X is *irreducible* (Cassandras and Lafortune, 2008).

We associate each cell with a non-zero, positive performance index that is a measure of the fitness of the cell, $F : X \to \mathbb{R}^+$. The notion of fitness facilitates the following novel mathematical definition of selection.

Definition 3. Given a cell set, X, and a fitness function $F : X \to \mathbb{R}^+$, let $Select : X \times X \times \mathbb{N} \to X$ be a random function such that if $x_1 \in X$ and $x_2 \in X$ are any two cells, and $N \in \mathbb{N}$ is the *level of selectivity*, then

$$Select(x_1, x_2, N) = \begin{cases} x_1 & \text{with probability } \dfrac{F(x_1)^N}{F(x_1)^N + F(x_2)^N}, \\ x_2 & \text{with probability } \dfrac{F(x_2)^N}{F(x_1)^N + F(x_2)^N}. \end{cases} \quad (4)$$

We can now define a selective evolutionary generation system (SEGS).

Definition 4. A *selective evolutionary generation system* is a quintuple
$\Gamma = (X, R, P, G, F)$, where

- (X, R, P, G) is an evolutionary generation system;
- $F : X \to \mathbb{R}^+$ is a function that evaluates cell fitness;
- the set of cells, X, is reachable through G and R; and
- the dynamics of the system are given by

$$\mathcal{X}(t+1) = Select(\mathcal{X}(t), G(\mathcal{X}(t), \mathcal{R}(t)), N). \quad (5)$$

In (5), $\mathcal{X}(t)$ denotes the realization of a random cell variable at time t, $\mathcal{R}(t)$ denotes the realization of a random resource variable at time t, $G(\mathcal{X}(t), \mathcal{R}(t))$ denotes the offspring of the realized random cell utilizing the realized random resource at time t, and $\mathcal{X}(0)$ has a known probability mass function.

Also in (5), the probability of a cell realization at some future time given the present cell realization is conditionally independent of the past time history of cell realizations. Thus, the dynamics of a SEGS form a discrete-time homogeneous Markov chain (Brémaud, 1999). This property is useful for the SEGS analysis conducted in Section 4.3.

The *Select* function has a number of interesting properties:

- For all N,
$$\frac{\Pr[Select(x_1, x_2, N) = x_1]}{\Pr[Select(x_1, x_2, N) = x_2]} = \left(\frac{F(x_1)}{F(x_2)}\right)^N. \quad (6)$$
That is, the ratio of the probabilities of selecting any two cells is equal to the ratio of their respective fitnesses raised to the power N. This property is called *local rationality.*
- For $N = 0$, the values of $F(x_1)$ and $F(x_2)$ are irrelevant. That is,
$$\Pr[Select(x_1, x_2, 0) = x_1] = 1/2, \text{ and} \quad (7)$$
$$\Pr[Select(x_1, x_2, 0) = x_2] = 1/2. \quad (8)$$
- When $N \to \infty$, if $F(x_1) > F(x_2)$ then
$$\Pr[Select(x_1, x_2, N) = x_1] \to 1. \quad (9)$$
On the other hand, if $F(x_1) < F(x_2)$ then
$$\Pr[Select(x_1, x_2, N) = x_2] \to 1. \quad (10)$$
- If $F(x_1) = F(x_2)$ then, for all N,
$$\Pr[Select(x_1, x_2, N) = x_1] = 1/2, \text{ and} \quad (11)$$
$$\Pr[Select(x_1, x_2, N) = x_2] = 1/2. \quad (12)$$

Comparisons between various optimization methodologies and a SEGS approach can be made by quantifying the ratio of the probability of selecting a candidate optimizer of the objective function to the probability of selecting the optimizer's offspring (see Menezes (2010)). The canonical genetic algorithm with fitness proportional selection and the (1+1) evolutionary strategy are particular cases of a SEGS scheme.

4. MARKOV CHAIN ANALYSIS OF SEGS

4.1 Efficiency and Goodness

Let $(X, \mathbf{P})$ be a time-homogeneous, irreducible, ergodic Markov chain, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of states of a Markov process, $\mathbf{P} \in \mathbb{R}^{n \times n}$ is the matrix of transition probabilities for these states, and $n < \infty$ is the number of states. Assume that the initial probability distribution over the states is known, i.e., we are given an n-vector $\mathbf{p}(0)$ having elements $p_i(0) = \Pr[\mathcal{X}(0) = x_i]$ for all $x_i \in X$, where $\mathcal{X}(0)$ denotes the state realization at time 0, and we have $\sum_{i=1}^{n} p_i(0) = 1$. Since we have assumed that the states in X are ergodic and irreducible, they admit a unique stationary probability distribution (Brémaud, 1999; Cassandras and Lafortune, 2008). Let $\boldsymbol{\pi} = [\pi_1 \ \pi_2 \ \ldots \ \pi_n]$ be the row vector of these stationary probabilities, satisfying the constraints $\pi_i > 0 \ \forall i$, and $\sum_{i=1}^{n} \pi_i = 1$. Let $F : X \to \mathbb{R}^+$ be a positive fitness function. Let $N \in \mathbb{N}$ be a natural number. We define rational behavior for this Markov chain as follows.

Definition 5. The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is said to *behave rationally* with respect to fitness F with level N if

$$\frac{\pi_i}{\pi_j} = \left(\frac{F(x_i)}{F(x_j)}\right)^N, \ 1 \le i \le n, \ 1 \le j \le n. \quad (13)$$

This is a definition of *global rationality.*

Each stationary probability can also be explicitly characterized to ensure Markov chain rational behavior, as is indicated by the following theorem.

Theorem 6. The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ behaves rationally with respect to fitness F with level N if and only if

$$\pi_i = \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N},\ 1 \le i \le n. \tag{14}$$

Proof. See Menezes (2010).

Here, we have a more general, probabilistic version of the optimization of an objective function. A Markov chain that behaves rationally selects the state of maximum fitness with the highest stationary probability, and, in the limit as N approaches ∞, this probability is 1. The problem and solution then revert to one of standard optimization. Remarkably, rational behavior in Markov chains is the result of a subsidiary optimization.

Theorem 7. The stationary distribution $\boldsymbol{\pi}$ of the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N solves the optimization problem

$$\min_{\pi_1,\ldots,\pi_n} U(\boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \tag{15}$$

subject to the constraints $\sum_{i=1}^{n} \pi_i = 1$, and $\pi_i > 0$, $\forall i$, utilizing the fitness distribution

$$\varphi_i = \frac{F(x_i)^N}{\sum_{k=1}^{n} F(x_k)^N},\ 1 \le i \le n. \tag{16}$$

Proof. See Menezes (2010).

Furthermore, Theorem 7 states that at the optimum, the stationary distribution agrees with the fitness distribution, i.e., $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Using the notion of entropy, we can interpret (15) as follows. First, we recognize the term $-\ln(\pi_i)$ as the information content of state x_i (Shannon, 1948). Hence, the right hand side of (15) represents the "fitness-expectation of information." Moreover, we have the following.

Corollary 8. The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ behaves rationally with respect to fitness F with level N if and only if its stationary probability distribution minimizes the fitness-expectation of information. At the optimum, this fitness-expectation of information is the entropy of the fitness distribution, i.e.,

$$U^* = H(\boldsymbol{\varphi}) = -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i). \tag{17}$$

Entropy maximization is important for search according to Jaynes (1957). The relationship between entropy maximization and optimal, efficient search is clarified in Jaynes (1981). Applying the results from Jaynes (1981) and Jaynes (1957), an exponential normalized fitness function relates rational behavior, entropy and efficient search through the following theorem.

Theorem 9. Let $y : X \to \mathbb{R}$ be an unknown function for which an expected value, $\mathrm{E}[y(x)]$, is a known number Y. The normalized fitness

$$\varphi_i = \alpha e^{-\beta y(x_i)},\ 1 \le i \le n, \tag{18}$$

and the stationary distribution $\boldsymbol{\pi}$ of the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N solves the optimization problem

$$\max_{\varphi_1,\ldots,\varphi_n} \min_{\pi_1,\ldots,\pi_n} U(\boldsymbol{\varphi}, \boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \tag{19}$$

subject to the constraint

$$\mathrm{E}[y(x)] = Y. \tag{20}$$

Proof. See Menezes (2010).

Hence, a scheme with underlying Markov chain dynamics that behave rationally also maximizes the entropy of the fitness distribution when the fitness function is exponential. The implication is that a fitness function like

$$F(x_i) = e^{-\left((z(x_i)-z_{des})^2\right)} \tag{21}$$

together with a scheme that makes use of rational behavior (e.g., SEGS) guarantees "good" behaviors efficiently.

4.2 Responsiveness

Definition 10. For any time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ with a positive fitness function for all the states in X, the *extrinsic resilience* of state x_i to changes in the fitness of state x_j, $j \ne i$, is defined as

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)}, \tag{22}$$

and the *intrinsic resilience* of state x_i to changes in its own fitness is taken to be

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)}. \tag{23}$$

Since the stationary distribution $\boldsymbol{\pi}$ has the closed form expression (14) for the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N, the extrinsic and intrinsic resiliencies are

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)} = \frac{-N\pi_i\pi_j}{F(x_j)},\ \forall j \ne i, \tag{24}$$

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)} = \frac{N\pi_i(1-\pi_i)}{F(x_i)}. \tag{25}$$

We say that the Markov chain $(X, \mathbf{P})$ is *resilient* if $\rho_{ij} \ne 0$ for all i and j.

The level of selectivity has the following asymptotic effect.

Theorem 11. For the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N,

$$\rho_{ij}\Big|_{\substack{N=0\\ j\ne i}} = \rho_{ii}\Big|_{N=0} = 0, \tag{26}$$

and

$$\lim_{\substack{N\to\infty\\ j\ne i}} \rho_{ij} = \lim_{N\to\infty} \rho_{ii} = 0. \tag{27}$$

Proof. See Menezes (2010).

As a result of Theorem 11, we have quantification that standard optimization ($N \to \infty$) is non-resilient.

Resilience is a result of Markov chain rational behavior:

Theorem 12. The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is resilient if the chain behaves rationally.

Proof. See Menezes (2010).

Resilience does not always imply Markov chain rational behavior (see Menezes (2010)). But:

Theorem 13. Ergodicity is a necessary condition for the time-homogeneous, irreducible Markov chain $(X, \mathbf{P})$ to be resilient.

Proof. See Menezes (2010).

4.3 SEGS as Markov Chains That Behave Rationally

Definition 14. Let $\Gamma = (X, R, P, G, F)$ be a SEGS. Let $x_i, x_j \in X$ be any two cells, and $r_k \in R$ be a resource. The *descendancy tensor*, δ, has elements

$$\delta_{ijk} = \begin{cases} 1 & \text{if } x_j = G(x_i, r_k), \\ & 1 \le i \le n,\ 1 \le j \le n,\ 1 \le k \le m, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Hence, the descendancy tensor indicates whether it is possible to produce cell x_j in one step from cell x_i, using resource r_k. We can use this tensor to create a matrix that represents the conditional probability of generating x_j given that the progenitor is x_i, by utilizing the probability of selecting each available resource and summing over all m resources as follows.

Definition 15. For the SEGS $\Gamma = (X, R, P, G, F)$, the *matrix of generation probabilities*, $\boldsymbol{\gamma}$, also called the unselective matrix of transition probabilities, has elements

$$\gamma_{ij} = \Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i], \tag{29}$$

$$= \sum_{k=1}^{m} \delta_{ijk} p_k,\ 1 \le i \le n,\ 1 \le j \le n. \tag{30}$$

This matrix is a stochastic matrix (see Menezes (2010)).

Recall that a SEGS follows the stochastic Markov process described by (5). Therefore, we can find a matrix of transition probabilities to describe the cell-to-cell transitions that occur as a result of the selection dynamics. For the SEGS $\Gamma = (X, R, P, G, F)$, the *matrix of transition probabilities*, $\mathbf{P}$, has elements

$$P_{ij} = \Pr[\mathcal{X}(t+1) = x_j \mid \mathcal{X}(t) = x_i], \tag{31}$$

$$= \Pr[Select(x_i, x_j, N) = x_j \mid \mathcal{X}(t) = x_i] \times \Pr[\text{offspring is } x_j \mid \text{progenitor is } x_i] \tag{32}$$

$$= \begin{cases} \dfrac{1}{1 + \left(\frac{F(x_i)}{F(x_j)}\right)^N} \gamma_{ij}, & \forall j \ne i, \\ \gamma_{ii} + \displaystyle\sum_{\substack{j=1 \\ j \ne i}}^{n} \dfrac{1}{1 + \left(\frac{F(x_j)}{F(x_i)}\right)^N} \gamma_{ij}, & \text{if } j = i. \end{cases} \tag{33}$$

Note that the matrix of transition probabilities in (33) is also a stochastic matrix (see Menezes (2010)).

Theorem 16. For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, assume that the matrix of generation probabilities, $\boldsymbol{\gamma}$, is symmetric. Then the Markov chain representing the stochastic dynamics of the ergodic SEGS behaves rationally with fitness F and level N. That is, the row vector $\boldsymbol{\pi} = [\pi_1\ \pi_2\ \dots\ \pi_n]$, where π_i satisfies (14), is a left eigenvector of $\mathbf{P}$, the matrix of transition probabilities for Γ, with corresponding eigenvalue 1 (i.e., $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$). Hence, $\boldsymbol{\pi}$ is the vector of stationary probabilities for the SEGS.

Proof. See Menezes (2010).

As a result of Theorem 12, the stochastic dynamics of the ergodic SEGS with symmetric matrix of generation probabilities, $\boldsymbol{\gamma}$, are resilient. Hence, a SEGS is a computationally inexpensive on-line technique to achieve these characteristics because only local decisions between two candidate optimizers are made at any time. The need to evaluate the fitness of all elements in the domain of the objective function, or even in a sub-population of candidate optimizers (as in genetic algorithms or evolutionary strategies), is avoided.

The symmetry condition on the matrix of generation probabilities, $\boldsymbol{\gamma}$, implies that there exists equiprobable forward and reverse transitions between any pair of cells prior to the selection process. More specifically, symmetry of $\boldsymbol{\gamma}$ is a requirement that mutations be reversible.

Theorem 17. For the ergodic SEGS $\Gamma = (X, R, P, G, F)$, assume that the matrix of generation probabilities, $\boldsymbol{\gamma}$, is symmetric. Then the Markov chain representing the stochastic dynamics of the ergodic SEGS is time-reversible, i.e.,

$$\pi_i P_{ij} = \pi_j P_{ji},\ \forall i, j. \tag{34}$$

Proof. See Menezes (2010).

As a consequence, the Markov chain representing the stochastic dynamics of the SEGS and its time reversed form are statistically the same.

4.4 Relationship Between SEGS and Markov Chain Monte Carlo Algorithms

The SEGS algorithm is an example of a Markov chain Monte Carlo (MCMC) algorithm. Since convergence to the target distribution, the stationary distribution $\boldsymbol{\pi}$, is easier to check for reversible Markov chains, these Markov chains are the most frequent case of MCMC algorithms (Brémaud, 1999). Hence, the design of an MCMC algorithm involves finding an ergodic transition matrix $\mathbf{P}$ that satisfies

$$\pi_i P_{ij} = \pi_j P_{ji},\ \forall i, j. \tag{35}$$

According to Brémaud (1999), a typical choice of P_{ij} has the form

$$P_{ij} = Q_{ij} \alpha_{ij},\ \forall j \ne i. \tag{36}$$

Here, $\mathbf{Q}$ is a probability transition matrix (called the *candidate-generating matrix*) with elements Q_{ij} representing the probability of "tentatively" choosing a transition from i to j, and $\boldsymbol{\alpha}$ is a probability transition matrix with elements α_{ij} representing the probability of accepting that transition. A generic formulation for the acceptance probabilities is specified by the Hastings algorithm, which sets

$$\alpha_{ij} = \frac{s_{ij}}{1 + \frac{\pi_i Q_{ij}}{\pi_j Q_{ji}}}, \tag{37}$$

where s_{ij} are the elements of a symmetric matrix $\mathbf{S}$. Special cases of the Hastings algorithm include the Metropolis algorithm, which is used in simulated annealing, and Barker's algorithm.

The acceptance probability for Barker's algorithm sets $s_{ij} = 1$ in (37), so that

$$\alpha_{ij} = \frac{1}{1 + \left(\frac{\pi_i}{\pi_j}\right)\left(\frac{Q_{ij}}{Q_{ji}}\right)}. \tag{38}$$

In the case of purely random $\mathbf{Q}$, this becomes

$$\alpha_{ij} = \frac{1}{1 + \left(\frac{\pi_i}{\pi_j}\right)}. \tag{39}$$

A SEGS has $\mathbf{Q} = \boldsymbol{\gamma}$. For rational behavior, we impose a symmetry condition so that $Q_{ij} = Q_{ji}$. Setting $s_{ij} = 1$ in (37), the definition of rational behavior implies that the acceptance probability utilized by the SEGS algorithm is

$$\alpha_{ij} = \frac{1}{1 + \left(\frac{\pi_i}{\pi_j}\right)}. \tag{40}$$

Thus, the SEGS algorithm and Barker's algorithm are the same. However, this paper arrived at Barker's algorithm in a non-traditional manner, i.e., we did not assume time-reversibility and begin at Hasting's algorithm. Instead, we started with a self-reproducing process and selected according to local rationality. The aim was to achieve global rational behavior, thereby resulting in resilience. A required assumption was equiprobable forward and reverse transitions prior to selection. This assumption resulted in the SEGS algorithm being time-reversible. Whereas the Metropolis algorithm is optimal with respect to asymptotic variance in the class of Hastings algorithms with fixed candidate-generating matrix $\mathbf{Q}$ Brémaud (1999), Barker's algorithm is optimal with respect to search efficiency under the technical conditions specified in Theorem 9.

5. CONCLUSIONS

This paper has proposed a novel on-line behavior design strategy by demonstrating and utilizing the fact that resilience is guaranteed by rational behavior, the use of which is desirable because it can lead to a search that trades off prior information for search effort savings as quickly as possible. Illustrative applications of the strategy, associated computational results, and comparisons with traditional algorithms are available in Menezes (2010).

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