Contents

Chapter 1

MARKOV CHAIN RATIONAL BEHAVIOR

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Motivation: Traditional off-line global optimization is non-resilient and non-opportunistic. That is, traditional global optimization is unresponsive to small perturbations of the objective function that require a small or large change in the optimizer. On-line optimization methods that are more resilient and opportunistic than their off-line counterparts typically consist of the computationally expensive sequential repetition of off-line techniques. A novel approach to on-line global optimization is to utilize the theory of rational behavior to develop a technique that is resilient, opportunistic, and inexpensive.

Overview: This paper proves that decision processes with time-homogeneous, irreducible, ergodic Markov chain dynamics that satisfy the axioms of rational behavior result in the resilient and opportunistic determination of a global optimizer for a given objective function. The optimization of control gains for a sample dynamic system illustrates the theory.

1.1 Introduction

The Theory of Rational Behavior (TRB), as described in Meerkov (1979), proposes an axiomatic behavior of individual elements that take decisions in a decision space. This theory captures a trait frequently observed in nature: individuals are capable of selecting the most favorable decision from among all possible options without executing complex calculations. The work seeks the computationally simplest mechanisms that lead to rationality (see also Lin (2006)).

More specifically, TRB defines general dynamical systems with certain decision spaces to be rational if the trajectories of these systems in the spaces are ergodic (i.e., the trajectories explore all decisions in the decision space), and selective (i.e., the trajectories slow down in the vicinity of the most advantageous states). The primary benefit of this approach is the possibility of rapid convergence to the optimal state. It has been postulated that the use of TRB in optimization yields additional benefits – this paper proves that a TRB-based optimization scheme is *responsive* to changes that alter the optimal state in a dynamic system.

1.1.1 The Need For Responsiveness

In standard optimization, we are given a mapping from a set X into the reals,

$$F : X \to \mathbb{R}. \tag{1.1}$$

Although the exact function may not be known *a priori*, the value of this function can typically be evaluated for all x in X. We seek an element, x^* in X, that satisfies

$$F(x^*) \geq F(x),\ \forall x \in X, \tag{1.2}$$

subject to certain constraints. This is known as the *global maximization* problem for the objective function (1.1).

The optimization accomplished by traditional deterministic algorithms (Kuhn and Tucker (1951), Dennis and Schnabel (1996), Ortega and Rheinboldt (2000), Luenberger (2003), Boyd and Vandenberghe (2004)) and other randomized algorithms such as simulated annealing (Kirkpatrick et al. (1983), Corana et al. (1987)), genetic algorithms (Goldberg (1989), Davis (1991), Mitchell (1996)) and evolutionary strategies (Rechenberg (1971), Schwefel (1995), Beyer and Schwefel (2002), Fogel (2006)) is *off-line* (as defined by Atallah (1999)), since optimization of the unknown objective function (1.1) is carried out under the assumption of time-independence of the values of candidate optimizers, and the algorithms therefore have advance access to a complete and unchanging data set. However, off-line optimization strategies are *non-resilient* and *non-opportunistic*, as illustrated in Figure 1.1. That is, traditional global optimization is unresponsive to perturbations of the objective function applied after the optimizer x^* is implemented. This unresponsiveness holds when the optimizer changes as a differentiable function of the perturbation (see Figure 1.1 (a)) or as a non-differentiable function of the perturbation (see Figure 1.1 (b)). We refer to these two cases as non-resilient and non-opportunistic, respectively. We formally define resiliency and opportunism in Section 1.2.

Resiliency and opportunism in an optimization scheme are important because, in general, there are no guarantees that the value of a candidate optimizer is time-independent and hence the same as the initial objective function value. For the limiting case where such guarantees exist, the distribution of

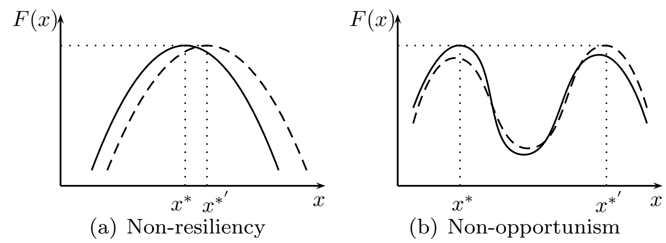

Figure 1.1: Off-line optimization strategies yield results that are non-resilient and non-opportunistic.

the candidate optimizers over the domain of the objective function is the delta function. In the more general case where perturbations of the objective function occur, a broader concept of optimization in the presence of perturbations is required. One expects that the distribution of candidate optimizers will now have an extended support.

The sequential repetition of off-line techniques results in *on-line* (with respect to the perturbations) optimization methods that are more resilient and opportunistic than their off-line counterparts. However, such sequential repetitions can be computationally expensive, a fact that may be shown by either an amortized analysis (Cormen et al. (2001)) or a competitive analysis (Borodin and El-Yaniv (1998)).

1.1.2 Goals and Contributions

The goal of this paper is to determine the conditions for which an on-line global optimization strategy is resilient and opportunistic for the standard optimization problem (1.2).

More specifically, this paper shows that rational behavior is a sufficient condition for resiliency and opportunism. The work then utilizes a rationally behaving decision process as a resilient and opportunistic on-line global scheme to optimize control gains for a sample dynamic system.

1.1.3 Chapter Outline

Section 1.2 demonstrates how time-homogeneous, irreducible, ergodic Markov chain dynamics that satisfy the axioms of rational behavior yield resiliency and opportunism. Section 1.3 illustrates the theory. Section 1.4 presents conclusions.

1.2 Markov Chains That Behave Rationally

In this section, we develop a Theory of Rational Behavior for time-homogeneous, irreducible, ergodic Markov chains. We then discuss the entropy, resiliency and opportunism of Markov chains that satisfy the axioms of this theory.

1.2.1 Markov Chain Rational Behavior

Let $(X, \mathbf{P})$ be a time-homogeneous, irreducible, ergodic Markov chain, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of states of a Markov process, $\mathbf{P} \in \mathbb{R}^{n \times n}$ is the matrix of transition probabilities for these states, and $n < \infty$ is the number of states. Assume that the initial probability distribution over the states is known, i.e., we are given an n-vector $\mathbf{p}(0)$ having elements $p_i(0) = \Pr[\mathcal{X}(0) = x_i]$ for all $x_i \in X$, where $\mathcal{X}(0)$ denotes the state realization at time 0, and we have $\sum\limits_{i=1}^{n} p_i(0) = 1$. Since we have assumed that the states in X are ergodic (i.e., positive recurrent and aperiodic) and irreducible, they admit a unique stationary probability distribution. Let $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_n \end{bmatrix}$ be the row vector of these stationary probabilities, satisfying the constraints $\pi_i > 0 \; \forall i$, and $\sum\limits_{i=1}^{n} \pi_i = 1$. Let $F : X \rightarrow \mathbb{R}^+$ be a positive *fitness* function for all the states. Let $N \in \mathbb{N}$ be a natural number. We define rational behavior for this Markov chain as follows.

Definition 1.1. *The time-homogeneous, irreducible, ergodic Markov chain* $(X, \mathbf{P})$ *is said to* behave rationally *with respect to fitness* F *with level* N *if*

$$\frac{\pi_i}{\pi_j} = \left(\frac{F(x_i)}{F(x_j)} \right)^N, \; 1 \leq i, j \leq n. \tag{1.3}$$

This definition is consistent with Meerkov (1979) because time averages and ensemble averages are equal in an ergodic process. The requirement that $\pi_i > 0 \; \forall i$ with $\sum\limits_{i=1}^{n} \pi_i = 1$ corresponds to the ergodic postulate of Meerkov (1979), and the requirement that $N > 0$ corresponds to the selective (i.e., retardation) postulate. Note that we have recast the requisite scalar function of Meerkov (1979) as a reward, instead of a penalty.

Each stationary probability can also be explicitly characterized to ensure Markov chain rational behavior, as is indicated by the following theorem.

Theorem 1.1. *The time-homogeneous, irreducible, ergodic Markov chain* $(X, \mathbf{P})$ *behaves rationally with respect to fitness* F *with level* N *if and only if*

$$\pi_i = \frac{F(x_i)^N}{\sum\limits_{k=1}^{n} F(x_k)^N}, \; 1 \leq i \leq n. \tag{1.4}$$

Proof. See Appendix. □

Here, we have a more general, probabilistic version of the optimization in (1.1). A Markov chain that behaves rationally will select the state of maximum fitness with the highest stationary probability, and, in the limit as N approaches ∞, this probability is 1. The problem and solution then revert to one of standard optimization.

Furthermore, rational behavior in Markov chains is the result of a subsidiary optimization.

Theorem 1.2. *The stationary distribution* $\boldsymbol{\pi}$ *of the time-homogeneous, irreducible, ergodic Markov chain* $(X, \mathbf{P})$ *that behaves rationally with respect to fitness* F *with level* N *solves the optimization problem*

$$\min_{\pi_1,\ldots,\pi_n} \Phi(\boldsymbol{\pi}) = -\sum_{i=1}^{n} F(x_i)^N \ln(\pi_i), \tag{1.5}$$

subject to the constraints

$$\sum_{i=1}^{n} \pi_i = 1, \tag{1.6}$$

$$\pi_i > 0, \ \forall i. \tag{1.7}$$

Proof. See Appendix. □

Note that in (1.3), rational behavior is invariant under positive scaling of fitness. Hence, there is no loss of generality in assuming that the fitness function is normalized. Accordingly, let $\boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 & \varphi_2 & \ldots & \varphi_n \end{bmatrix}$ be the distribution of the N^{th} power of fitness, where

$$\varphi_i = \frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n} F\left(x_k\right)^N}, \ 1 \leq i \leq n. \tag{1.8}$$

The vector $\boldsymbol{\varphi} \in \mathbb{R}^n$ is a *distribution of order* n because it satisfies $\varphi_i > 0 \ \forall i$, and $\sum\limits_{k=1}^{n} \varphi_k = 1$. Let

$$U(\boldsymbol{\pi}) = \frac{\Phi(\boldsymbol{\pi})}{\sum\limits_{k=1}^{n} F\left(x_k\right)^N}. \tag{1.9}$$

Then, the optimization problem (1.5) can be normalized as

$$\min_{\pi_1,\ldots,\pi_n} U(\boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \tag{1.10}$$

subject to the constraints (1.6) and (1.7). Furthermore, Theorem 1.2 states that at the optimum, the stationary distribution agrees with the fitness distribution, i.e., $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

1.2.2 Entropy of Markov Chains That Behave Rationally

Let $\mathbb{D}_n$ be the set of distributions of order n, and define the entropy of a distribution (Shannon (1948)) as follows.

Definition 1.2. Entropy *is the function*

$$H : \mathbb{D}_n \to \mathbb{R} : \boldsymbol{\varphi} \mapsto H(\boldsymbol{\varphi}) = -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i). \tag{1.11}$$

Using the notion of entropy, we can interpret (1.10) as follows. First, we recognize the term $-\ln(\pi_i)$ as the information content of state x_i (Shannon (1948)). Hence, the right hand side of (1.10) represents the "fitness-expectation of information." Moreover, we have the following:

Corollary 1.1. *The time-homogeneous, irreducible, ergodic Markov chain* $(X, \mathbf{P})$ *behaves rationally with respect to fitness* F *with level* N *if and only if its stationary probability distribution minimizes the fitness-expectation of information. At the optimum, this fitness-expectation of information is the entropy of the fitness distribution, i.e.,*

$$U^* = H(\boldsymbol{\varphi}) = -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i). \tag{1.12}$$

A basic property of entropy that is alluded to in Kerridge (1961) is as follows.

Theorem 1.3. *Let* $\boldsymbol{\varphi} \in \mathbb{D}_n$ *be arbitrary. Then,*

$$\min_{\boldsymbol{\pi} \in \mathbb{D}_n} -\sum_{i=1}^{n} \varphi_i \ln(\pi_i), \tag{1.13}$$

has a minimum value of $H(\boldsymbol{\varphi})$ *that is achieved at* $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Equivalently, $\forall \boldsymbol{\varphi} \in \mathbb{D}_n$, $\forall \boldsymbol{\pi} \in \mathbb{D}_n$,

$$-\sum_{i=1}^{n} \varphi_i \ln(\pi_i) \geq -\sum_{i=1}^{n} \varphi_i \ln(\varphi_i), \tag{1.14}$$

with the equality holding if and only if $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Equivalently, $\forall \boldsymbol{\varphi} \in \mathbb{D}_n$, $\boldsymbol{\pi} \in \mathbb{D}_n$,

$$-\sum_{i=1}^{n} \varphi_i \ln\left(\frac{\pi_i}{\varphi_i}\right) \geq 0, \tag{1.15}$$

with the equality holding if and only if $\boldsymbol{\pi} = \boldsymbol{\varphi}$.

Proof. See Appendix. □

For Markov chains that behave rationally, and therefore possess fitness fractions that are distributed over the set of states as in (1.8), the entropy quantifies how egalitarian or elitist the states are. That is, the entropy is highest when all states have equal fitness; conversely, the entropy is lowest when there is only one state with a fitness fraction of unity and all other fitness fractions are zero. Equation (1.11) arises in other well-known fields, and similar interpretations for the distributed quantities and the entropy exist (Shannon (1948), Kerridge (1961), Pathria (1996), Cengel and Boles (2001)).

1.2.3 Resiliency and Opportunism of Markov Chains That Behave Rationally

We can now formally define resiliency, first described through Figure 1.1 (a), as the sensitivity of the stationary distribution to changes in fitness.

Definition 1.3. *For any time-homogeneous, irreducible, aperiodic Markov chain* $(X, \mathbf{P})$ *with a positive fitness function for all the states in* X*, the* extrinsic resiliency *of state* x_i *to changes in the fitness of state* x_j*,* $j \neq i$*, is defined as*

$$\rho_{ij} = \frac{\partial \pi_i}{\partial F(x_j)}, \tag{1.16}$$

and the intrinsic resiliency *of state* x_i *to changes in its own fitness is taken to be*

$$\rho_{ii} = \frac{\partial \pi_i}{\partial F(x_i)}. \tag{1.17}$$

Since the stationary distribution $\boldsymbol{\pi}$ has the closed form expression (1.4) for the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N, the extrinsic and intrinsic resiliencies become

$$\begin{aligned} \rho_{ij} &= \frac{\partial \pi_i}{\partial F(x_j)} = \frac{-N\pi_i\pi_j}{F(x_j)}, \ \forall j \neq i, && (1.18)\\ \rho_{ii} &= \frac{\partial \pi_i}{\partial F(x_i)} = \frac{N\pi_i\left(1-\pi_i\right)}{F(x_i)}. && (1.19) \end{aligned}$$

We say that the Markov chain $(X, \mathbf{P})$ is *resilient* if $\rho_{ij} \neq 0$ for all i and j.

Implicit in resiliency equations (1.16) and (1.17) is the differentiability of the function $\pi_i\left(F\left(x_1\right), \ldots, F\left(x_n\right)\right)$ as given by (1.4). If $\pi_i\left(F\left(x_1\right), \ldots, F\left(x_n\right)\right)$ is not differentiable, as in Figure 1.1 (b), then we can define opportunism through the use of the more general Gâteaux derivative, if it exists. Thus, using the notation of Luenberger (1969), the *extrinsic opportunism* of state x_i to changes in the fitness of state x_j, $j \neq i$, is defined as

$$\omega_{ij} = d_j\pi_i(\mathbf{F}; \mathbf{e}_j), \tag{1.20}$$

and the *intrinsic opportunism* of state x_i to changes in its own fitness is taken to be

$$\omega_{ii} = d_i\pi_i(\mathbf{F}; \mathbf{e}_i), \tag{1.21}$$

where $\mathbf{F}$ is the vector of fitness values $\begin{bmatrix} F(x_1) & \ldots & F(x_n) \end{bmatrix}$, and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$. We say that the Markov chain $(X, \mathbf{P})$ is *opportunistic* if $\omega_{ij} \neq 0$ for all i and j.

For Markov chains that behave rationally, the definitions of resiliency and opportunism are the same because of the equivalence of the Gâteaux and partial derivatives.

The level of selectivity has the following asymptotic effect on resiliency.

Theorem 1.4. *For the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ that behaves rationally with respect to fitness F with level N,*

$$\rho_{ij}\Big|_{N=0} = \rho_{ii}\Big|_{N=0} = 0, \tag{1.22}$$

and

$$\lim_{N\to\infty} \rho_{ij} = \lim_{N\to\infty} \rho_{ii} = 0. \tag{1.23}$$

Proof. See Appendix. □

As a result of Theorem 1.4, we have quantification that standard optimization ($N = \infty$) is non-resilient.

Resiliency and opportunism is a direct outcome of Markov chain rational behavior, as stated below.

Theorem 1.5. *The time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is resilient and opportunistic if the chain behaves rationally.*

Proof. See Appendix. □

Resiliency and opportunism do not always imply Markov chain rational behavior. But we can state the following instead.

Theorem 1.6. *Ergodicity is a necessary condition for the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ to be resilient and opportunistic.*

Proof. See Appendix. □

Furthermore, there is a fundamental trade-off between extrinsic and intrinsic resiliency that is imposed by the constraint $\sum_{i=1}^{n} \pi_i = 1$. Taking the partial derivative of this constraint with respect to the fitness of state x_i, we obtain

$$\frac{\partial \pi_i}{\partial F(x_i)} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \pi_j}{\partial F(x_i)} = 0. \tag{1.24}$$

The implication is that any change in fitness that improves a state's intrinsic resiliency is at the expense of the extrinsic resiliency of all other states. Similarly, any change in fitness that improves a state's extrinsic resiliency is at the expense of the intrinsic resiliency of another state, and the extrinsic resiliency of all other states.

1.3 Example Application

This section applies the theory developed in Section 1.2 to a model of a dynamic system that Professor Meerkov and the authors are interested in. This model was developed with collaborators in industry, and refers to a state-of-the-art technological system. Utilizing a scheme that is developed in Menezes and Kabamba (2009), which is a decision process with time-homogeneous, irreducible, ergodic Markov chain dynamics that satisfy the axioms of rational behavior, the problem is to optimize control gains such that acceptable system performance is achieved.

1.3.1 Problem Description

Consider the system block diagram in Figure 1.2, where the plants, P_1 and P_2, and the plant output, v_3, are subject to external disturbances $d_1 \in [d_{1i}, d_{1f}]$ and $d_2 \in [d_{2i}, d_{2f}]$ as shown. The input r is a reference signal, v_1, v_2 and v_3 are intermediate signals, and y is the output signal. Control signals u_1 and u_2 utilize the measured signals v_2 and y, and control gains K_1 and K_2.

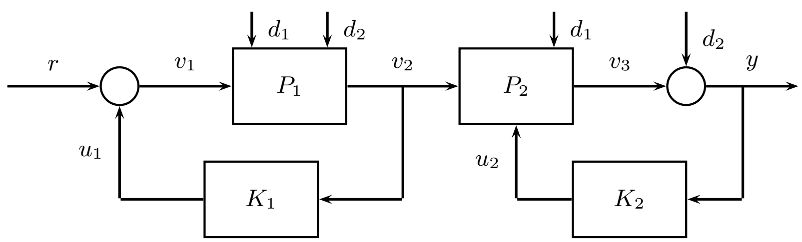

Figure 1.2: Block diagram of the example dynamic system.

Let the set of states, X, be the set of ordered pairs (K_1, K_2) where K_1 and K_2 take discrete values over a finite interval, e.g., $-20 \leq K_1, K_2 \leq 20$. Since the desired output is $y_{des} = 0$, with an acceptable tolerance of ± 1, a suitable fitness function for the states is

$$F = \exp\left(-\left(y_{des} - y\right)^2\right). \tag{1.25}$$

Transitions between states take place in accordance with Menezes and Kabamba (2009), in such a way that a random walk from one (K_1, K_2) pair to another transitions according to the ratio of fitnesses of the states raised to the power of N.

1.3.2 Results

A sample run of the optimization scheme when $N = 5$ is depicted in Figures 1.3 to 1.6 for fixed disturbances d_1 and d_2. A pair of control gains that achieves satisfactory performance is found within 50 generations. To demonstrate disturbance rejection, the disturbances are varied after 50 generations and the scheme is quickly able to find a new pair of gains that achieves an acceptable output.

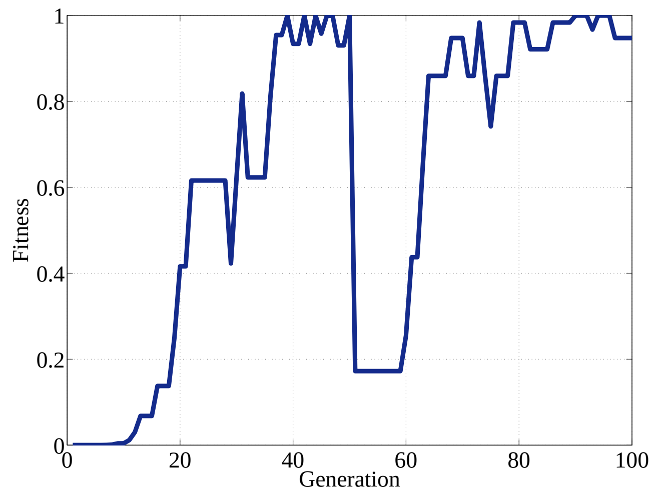

Figure 1.3: Fitness of the control gains per generation.

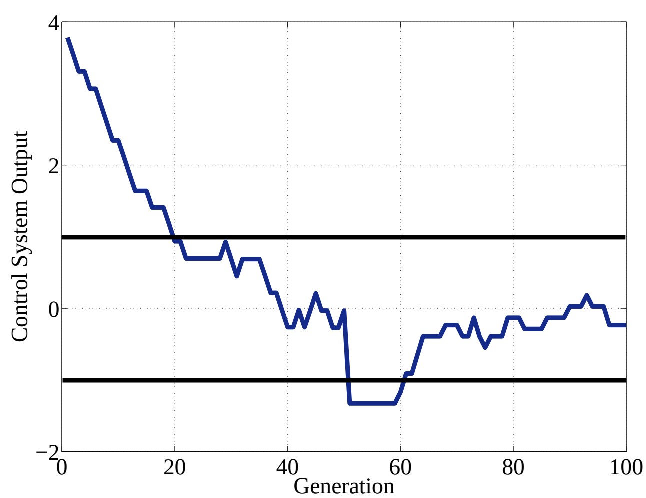

Figure 1.4: Satisfactory output is maintained despite disturbance changes at generation 50.

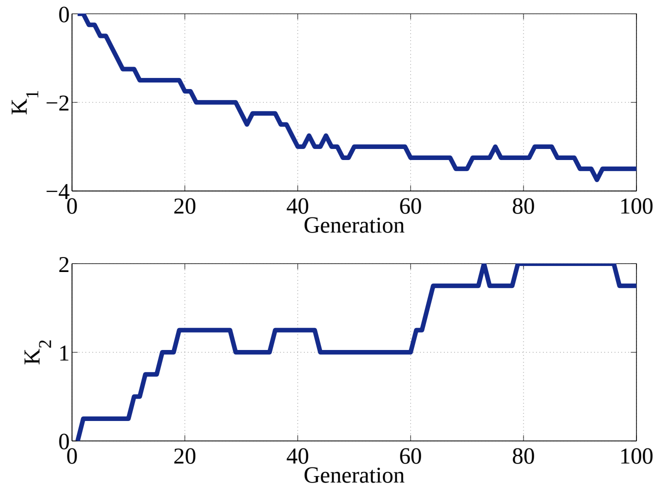

Figure 1.5: Control gain pairs per generation.

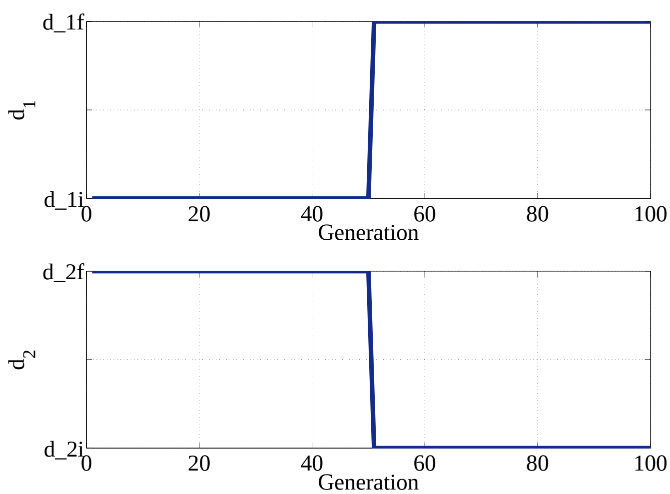

Figure 1.6: Disturbance variations at generation 50.

Similarly, the optimization scheme is resilient to internal model variations, and this is depicted in Figures 1.7 to 1.9 for fixed disturbances d_1 and d_2, $N = 5$, and an internal parameter change at the 50th generation.

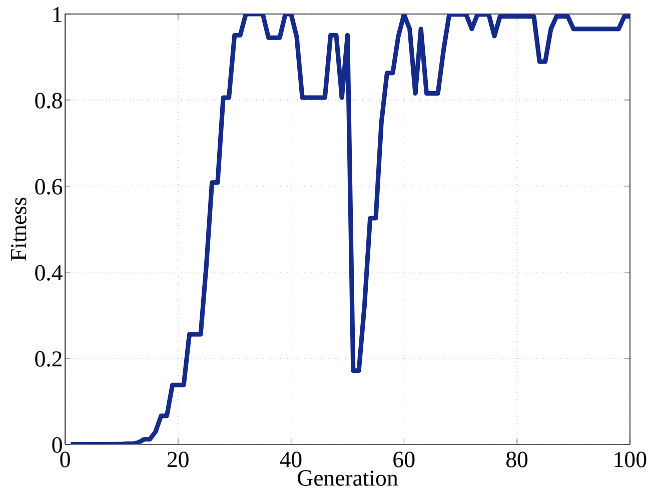

Figure 1.7: Fitness of the control gains per generation.

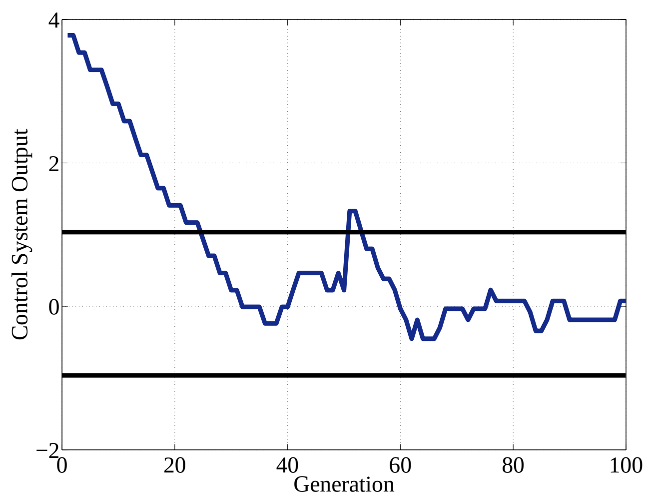

Figure 1.8: Satisfactory output is maintained despite an internal parameter change at generation 50.

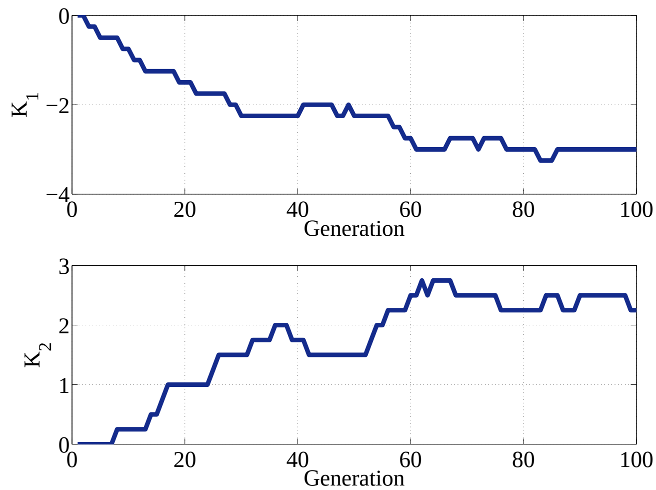

Figure 1.9: Control gain pairs per generation.

Typically, the scheme averages 21 seconds to compute the output of 100 generations while running in MATLAB on a 1.4 GHz single processor desktop computer with 1 GB of RAM.

1.4 Summary

This paper has

- demonstrated that the desirable characteristics of resiliency and opportunism in a time-homogeneous, irreducible, ergodic Markov chain are guaranteed by rational behavior. The ratio of the stationary probability of the optimizer of a fitness function to any other element's stationary probability is given by

$$\frac{\pi_I}{\pi_j} = \left(\frac{F(x_I)}{F(x_j)}\right)^N, \ 1 \leq j \leq n, \tag{1.26}$$

  where $F(x_I) > F(x_i)$ for all i implies that x_I is the most frequent. In the limit as N approaches infinity, π_I approaches 1, and standard optimization is recovered.

- shown that the entropy quantifies how egalitarian or elitist the time-homogeneous, irreducible, ergodic Markov chain rational behavior is.

- indicated that resiliency is a conserved quantity, and any improvements to the resiliency of a particular element decreases the resiliency of the other elements.

- utilized the theory to successfully optimize the control gains for a sample dynamic system, without expending significant computation effort.

1.5 Appendix

Theorem 1.1.

Proof. To show that (1.4) implies Markov chain rational behavior, consider the ratio of any π_i to π_j, $i \neq j$, where each satisfies (1.4). Equation (1.3) follows immediately.

To show that Markov chain rational behavior implies (1.4), we begin with

$$\sum_{k=1}^{n} \pi_k = 1.$$

Dividing both sides of the equation by π_i, we obtain

$$\sum_{k=1}^{n} \frac{\pi_k}{\pi_i} = \frac{1}{\pi_i},\ 1 \leq i \leq n,$$

which, using (1.3), yields

$$\sum_{k=1}^{n} \left(\frac{F(x_k)}{F(x_i)}\right)^N = \frac{1}{\pi_i},\ 1 \leq i \leq n.$$

Multiplying by $F(x_i)^N$ and solving for π_i yields (1.4), which completes the proof. $\square$

Theorem 1.2.

Proof. We use the method of Karush-Kuhn-Tucker (KKT) multipliers to solve the optimization problem

$$\min_{\pi_1,\ldots,\pi_n} \Phi(\boldsymbol{\pi}) = -\sum_{i=1}^{n} F(x_i)^N \ln(\pi_i),$$

subject to

$$\sum_{i=1}^{n} \pi_i - 1 = 0,$$

$$-\pi_i < 0,\ 1 \leq i \leq n.$$

Let $L(\pi_1, \ldots, \pi_n, \lambda, \mu_1, \ldots, \mu_n) =$

$$-\sum_{i=1}^{n} F(x_i)^N \ln(\pi_i) + \lambda \left(\sum_{i=1}^{n} \pi_i - 1\right) - \sum_{i=1}^{n} \mu_i \pi_i.$$

The KKT necessary conditions for optimality are

$$\frac{-F(x_i)^N}{\pi_i} + \lambda - \mu_i = 0,\ 1 \le i \le n,$$

$$\sum_{i=1}^{n} \pi_i - 1 = 0,$$

$$-\pi_i < 0,\ 1 \le i \le n,$$

$$\lambda \ge 0,$$

$$\mu_i \ge 0,\ 1 \le i \le n,$$

$$\lambda\left(\sum_{i=1}^{n} \pi_i - 1\right) = 0,$$

$$\mu_i \pi_i = 0,\ 1 \le i \le n.$$

The first necessary condition becomes

$$-F(x_i)^N + \lambda\pi_i - \mu_i\pi_i = 0,\ 1 \le i \le n.$$

Since $\mu_i\pi_i = 0$ for all i, we obtain

$$-F(x_i)^N + \lambda\pi_i = 0,\ 1 \le i \le n.$$

Next, the constraint $\pi_i > 0$ for all i and the positive nature of $F(x_i)^N$ imply that $\lambda \ne 0$. Therefore,

$$\pi_i = \frac{F(x_i)^N}{\lambda},\ 1 \le i \le n.$$

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \frac{F(x_i)^N}{\lambda},\ 1 \le i \le n.$$

Since $\sum\limits_{i=1}^{n} \pi_i = 1$, we find that

$$\lambda = \sum_{i=1}^{n} F(x_i)^N,$$

and hence,

$$\pi_i = \frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n} F\left(x_k\right)^N}\ ,\ 1 \le i \le n.$$

Thus, the stationary distribution in (1.4) satisfies the first order necessary conditions for optimality.

Moreover, we have

$$\frac{\partial^2 \Phi(\boldsymbol{\pi})}{\partial \pi_j \partial \pi_i} = 0 \text{ for } j \ne i,$$

$$\frac{\partial^2 \Phi(\boldsymbol{\pi})}{\partial \pi_i^2} = \frac{F(x_i)^N}{\pi_i^2} > 0.$$

Hence, the optimization problem has a strictly convex cost function and linear constraints. Thus, the solution of the first order necessary conditions is the global optimizer, which completes the proof. □

Theorem 1.3.

Proof. Similar to Theorem 1.2, we use the method of Karush-Kuhn-Tucker (KKT) multipliers to solve the following optimization problem for arbitrary $\boldsymbol{\varphi} \in \mathbb{D}_n$.

$$\min_{\boldsymbol{\pi} \in \mathbb{D}_n} - \sum_{i=1}^{n} \varphi_i \ln(\pi_i)$$

is equivalent to

$$\min_{\pi_1, \ldots, \pi_n} \Phi(\boldsymbol{\pi}) = -\sum_{i=1}^{n} \varphi_i \ln(\pi_i),$$

subject to

$$\sum_{i=1}^{n} \pi_i - 1 = 0,$$

$$-\pi_i < 0, \ 1 \leq i \leq n.$$

Let $L(\pi_1, \ldots, \pi_n, \lambda, \mu_1, \ldots, \mu_n) =$

$$-\sum_{i=1}^{n} \varphi_i \ln(\pi_i) + \lambda \left(\sum_{i=1}^{n} \pi_i - 1 \right) - \sum_{i=1}^{n} \mu_i \pi_i.$$

The KKT necessary conditions for optimality are

$$\frac{-\varphi_i}{\pi_i} + \lambda - \mu_i = 0, \ 1 \leq i \leq n,$$

$$\sum_{i=1}^{n} \pi_i - 1 = 0,$$

$$-\pi_i < 0, \ 1 \leq i \leq n,$$

$$\lambda \geq 0,$$

$$\mu_i \geq 0, \ 1 \leq i \leq n,$$

$$\lambda \left(\sum_{i=1}^{n} \pi_i - 1 \right) = 0,$$

$$\mu_i \pi_i = 0, \ 1 \leq i \leq n.$$

The first necessary condition becomes

$$-\varphi_i + \lambda \pi_i - \mu_i \pi_i = 0, \ 1 \leq i \leq n.$$

Since $\mu_i \pi_i = 0$ for all i, we obtain

$$-\varphi_i + \lambda \pi_i = 0,\ 1 \le i \le n.$$

Next, the constraint $\pi_i > 0$ for all i and the positive nature of φ_i imply that $\lambda \neq 0$. Therefore,

$$\pi_i = \frac{\varphi_i}{\lambda},\ 1 \le i \le n.$$

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \frac{\varphi_i}{\lambda},\ 1 \le i \le n.$$

Since $\sum\limits_{i=1}^{n} \pi_i = 1$, we find that

$$\lambda = \sum_{i=1}^{n} \varphi_i = 1$$

because $\boldsymbol{\varphi} \in \mathbb{D}_n$. Hence,

$$\pi_i = \varphi_i,\ 1 \le i \le n,$$

satisfies the first order necessary conditions for optimality. The minimum value is the entropy, (1.11).

Moreover, we have

$$\frac{\partial^2 \Phi(\boldsymbol{\pi})}{\partial \pi_j \partial \pi_i} = 0 \text{ for } j \neq i,$$

$$\frac{\partial^2 \Phi(\boldsymbol{\pi})}{\partial \pi_i^2} = \frac{\varphi_i}{\pi_i^2} > 0.$$

Hence, the optimization problem has a strictly convex cost function and linear constraints. Thus, the solution of the first order necessary conditions is the global optimizer, which completes the proof. □

Theorem 1.4.

Proof. We prove both parts of this theorem directly. Consider that

$$\rho_{ij}\Big|_{N=0} = \left.\frac{-N\pi_i\pi_j}{F(x_j)}\right|_{N=0},$$

$$= \left.\frac{-N}{F(x_j)} \frac{F(x_i)^N}{\sum\limits_{k=1}^{n} F(x_k)^N} \frac{F(x_j)^N}{\sum\limits_{k=1}^{n} F(x_k)^N}\right|_{N=0}.$$

By substitution, $\rho_{ij}\Big|_{N=0}$ is 0. Similarly,

$$\begin{aligned}\rho_{ii}\Big|_{N=0} &= \frac{N\pi_i\left(1-\pi_i\right)}{F(x_i)}\Bigg|_{N=0}, \\ &= \frac{N}{F(x_i)}\frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n}F\left(x_k\right)^N}\left(1-\frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n}F\left(x_k\right)^N}\right)\Bigg|_{N=0}.\end{aligned}$$

By substitution, $\rho_{ii}\Big|_{N=0}$ is also 0.

For the second part of the theorem, we need the following lemma.

Lemma 1.1. *Let $0<\alpha<1$. Then $\lim\limits_{N\to\infty} N\alpha^N = 0$.*

This lemma may be proved through the application of L'Hôpital's rule.

Let I be the index for which $F(x_i)$ is maximized, and assume that I is unique. Then,

$$\lim_{N\to\infty}\frac{F\left(x_j\right)^N}{F\left(x_I\right)^N}=0,\ \forall j\neq I,\ \text{and}$$

$$\lim_{N\to\infty}\sum_{k=1}^{n}\frac{F\left(x_k\right)^N}{F\left(x_I\right)^N}=1.$$

Consider that

$$\begin{aligned}\lim_{N\to\infty}\rho_{ij} &= \lim_{N\to 0}\frac{-N\pi_i\pi_j}{F(x_j)}, \\ &= \lim_{N\to\infty}\frac{-N}{F(x_j)}\frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n}F\left(x_k\right)^N}\frac{F\left(x_j\right)^N}{\sum\limits_{k=1}^{n}F\left(x_k\right)^N}, \\ &= \lim_{N\to\infty}\frac{-N}{F(x_j)}\frac{\frac{F(x_i)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n}\frac{F(x_k)^N}{F(x_I)^N}}\frac{\frac{F(x_j)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n}\frac{F(x_k)^N}{F(x_I)^N}}.\end{aligned}$$

Now for all $i\neq j$, where $i\neq I$ and $j\neq I$, the application of Lemma 1.1 with $\alpha=\frac{F(x_i)}{F(x_I)}$ implies that $\lim\limits_{N\to\infty}\rho_{ij}$ is equal to 0.

If $i=I\neq j$, then the application of Lemma 1.1 with $\alpha=\frac{F(x_j)}{F(x_I)}$ implies that $\lim\limits_{N\to\infty}\rho_{ij}$ is equal to 0.

Lastly, if $i\neq j=I$, then the application of Lemma 1.1 with $\alpha=\frac{F(x_i)}{F(x_I)}$ implies that $\lim\limits_{N\to\infty}\rho_{ij}$ is equal to 0.

Thus, for all i and j, $\lim\limits_{N\to\infty}\rho_{ij}=0$.

Similarly,

$$\lim_{N\to\infty}\rho_{ii} = \lim_{N\to 0}\frac{N\pi_i\left(1-\pi_i\right)}{F(x_i)},$$

$$= \lim_{N\to\infty}\frac{N}{F(x_i)}\frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n}F\left(x_k\right)^N}\left(1-\frac{F\left(x_i\right)^N}{\sum\limits_{k=1}^{n}F\left(x_k\right)^N}\right),$$

$$= \lim_{N\to\infty}\frac{N}{F(x_i)}\frac{\frac{F(x_i)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n}\frac{F(x_k)^N}{F(x_I)^N}}\frac{\sum\limits_{\substack{k=1\\k\neq i}}^{n}\frac{F(x_k)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n}\frac{F(x_k)^N}{F(x_I)^N}}.$$

If $i \neq I$, then the application of Lemma 1.1 with $\alpha = \frac{F(x_i)}{F(x_I)}$ implies that $\lim\limits_{N\to\infty}\rho_{ii}$ is equal to 0.

If $i = I$, then we have

$$\lim_{N\to\infty}\rho_{ii} = \lim_{N\to\infty}\frac{N}{F(x_I)}\frac{\frac{F(x_I)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n}\frac{F(x_k)^N}{F(x_I)^N}}\frac{\sum\limits_{\substack{k=1\\k\neq I}}^{n}\frac{F(x_k)^N}{F(x_I)^N}}{\sum\limits_{k=1}^{n}\frac{F(x_k)^N}{F(x_I)^N}}.$$

The application of Lemma 1.1 with $\alpha = \frac{F(x_k)}{F(x_I)}$ a total of $n-1$ times implies that $\lim\limits_{N\to\infty}\rho_{ii}$ is equal to 0.

Thus, for all i, $\lim\limits_{N\to\infty}\rho_{ii} = 0$. This completes the proof. □

Theorem 1.5.

Proof. To show that rational behavior implies that the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ is resilient, consider (1.18) and (1.19), which hold because the stationary distribution $\boldsymbol{\pi}$ has the closed form expression (1.4). By Definition 1.1, $\pi_i > 0\ \forall i$ since the Markov chain is ergodic, $N > 0$ since the Markov chain is selective, and $F(x_i) > 0\ \forall i$ since the fitness function is positive. Hence, $\rho_{ij} \neq 0\ \forall i$ and j, and $(X, \mathbf{P})$ is resilient. Since $\omega_{ij} = \rho_{ij} \neq 0\ \forall i$ and j, $(X, \mathbf{P})$ is also opportunistic. This completes the proof. □

Theorem 1.6.

Proof. To show that ergodicity is a necessary condition for the time-homogeneous, irreducible, ergodic Markov chain $(X, \mathbf{P})$ to be resilient and opportunistic, suppose that the chain is not ergodic. Then the chain is either not positive recurrent (i.e., it is null recurrent or transient) or it is periodic. If the chain is not positive recurrent, then there exists a zero stationary probability for a state, x_i. Suppose now that the fitness function is perturbed such that the fitness of this state, $F(x_i)$, becomes the optimal fitness value. Since the stationary probability of x_i is zero, element x_i is never visited, and therefore never considered as the optimizer. We have $\rho_{ii} = \partial\pi_i/\partial F(x_i) = 0$, and hence $(X, \mathbf{P})$ is not resilient. If the chain is periodic, then the stationary probability distribution does not exist, and neither resiliency nor opportunism is defined. This completes the proof. $\square$

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